

Four Metrics*

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Abstract

A central idea in general relativity is that physics should not depend on the space-time coordinates in use [1]. But the qualitative description of various phenomena can appear superficially quite different. Here we consider falling into a classical black hole using four distinct but equivalent metrics. First is the Schwarzschild case, with extreme time dilation at the horizon. Second, rescaling the dilation allows falling into the hole in finite proper time. Third, time and space are rescaled into a Penrose motivated picture where light trajectories all have unit slope. Fourth, a white hole variation of the second metric allows passage out through the horizon, with reentry forbidden.

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1 The Schwarzschild metric

We start with the famous Schwarzschild [2] formula for space-time around a stationary and non-rotating gravitating object

$$ds^2 = (1 - 1/r)dt^2 - (1 - 1/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (1)$$

Here we use polar coordinates for the spatial components

$$x = r \sin(\theta) \cos(\phi), \quad y = r \sin(\theta) \sin(\phi), \quad z = r \cos(\theta).$$

For simplicity, measure distances in units of the “Schwarzschild radius,” *i.e.* the radius of the horizon is taken as unity.

Light trajectories are defined by $ds^2 = 0$. For radial motion we have

$$\frac{dt}{dx} = \pm(1 - 1/r)^{-1}. \quad (2)$$

The plus (minus) sign is for an out-going (in-going) wave. This integrates to

$$t - t_0 = \pm \left(r - r_0 + \log \left(\frac{r - 1}{r_0 - 1} \right) \right). \quad (3)$$

Here r_0 and t_0 locate the initial point for the trajectory.

Remarkably as r approaches unity for an ingoing wave, time goes to infinity! The light never reaches the horizon. With this metric, neither light can leave the black hole, nor can it enter it. As nothing goes faster than light, nothing else can reach the horizon in finite Schwarzschild time. This behavior is sketched in Fig. 1. The figure also shows the path of a freely falling object initially at rest at $\{t, r\} = \{0, 2\}$. With the falling trajectory, tick marks are indicated at intervals of 0.1 in proper time s as measured on the corresponding path. Note how these marks spread as the time variable t increases. Time dilates as the black hole is approached. The spread increases rapidly with time, and $t = \infty$ is reached at finite proper time.

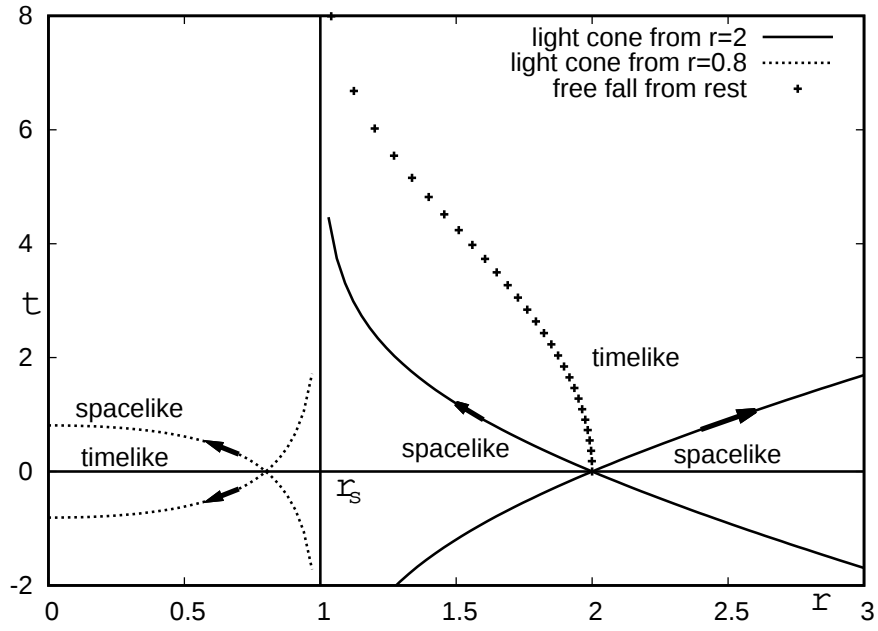


Figure 1: Light cones in Schwarzschild coordinates in the vicinity of a black hole. The radius of the horizon is taken as unity. One cone shown starts outside the hole at radius 2. The second starts at radius 0.8, inside the black hole. Note how the cone inside the hole is turned sideways relative to the outer one. The figure also shows the trajectory of a freely falling object started at $r = 2$, with tick marks indicating constant intervals in its proper time.

2 Falling through the horizon

As is well known, the singularity in the Schwarzschild metric at the horizon is somewhat artificial. As first made clear by Lemaitre [3], we can smooth the singularity at the horizon with an appropriate redefinition of coordinates.

For a simple example, consider a new definition of time, call it w , obtained by a change of variables similar to that made by Finkelstein [4]

$$w = t + \log(1 - 1/r). \quad (4)$$

The addition is singular at the horizon, $r = 1$, but this serves to remove the unphysical singularity in the Schwarzschild metric. The metric equation now becomes

$$ds^2 = dw^2 (1 - 1/r) - 2dw dr/r^2 - dr^2 (1 + 1/r)(1 + 1/r^2) - r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (5)$$

This is smooth at the horizon. The coordinate w can properly be considered as an alternate “time” in the sense that constant w surfaces continue to have negative ds^2 ; *i.e.* they remain space-like. The horizon is still well defined as a “separatrix,” dividing “out-going” light-waves into those that ultimately head towards $r = 0$ or $r = \infty$. This behavior is sketched in Fig. 2.

This figure is not symmetric when inverting the time coordinate w . Any time translation invariant coordinate system allowing crossing of the horizon is necessarily not time reversal symmetric. The time reversed black hole is what is sometimes called a “white hole,” to which we return later.

As with the Schwarzschild coordinates, this figure also shows the trajectory for a freely falling mass starting at radius 2. Again, this is marked with tick marks at constant separation in proper time. Now the gravitational red shift continues to increase even inside the hole, with the object reaching $w = \infty$ in a finite proper time.

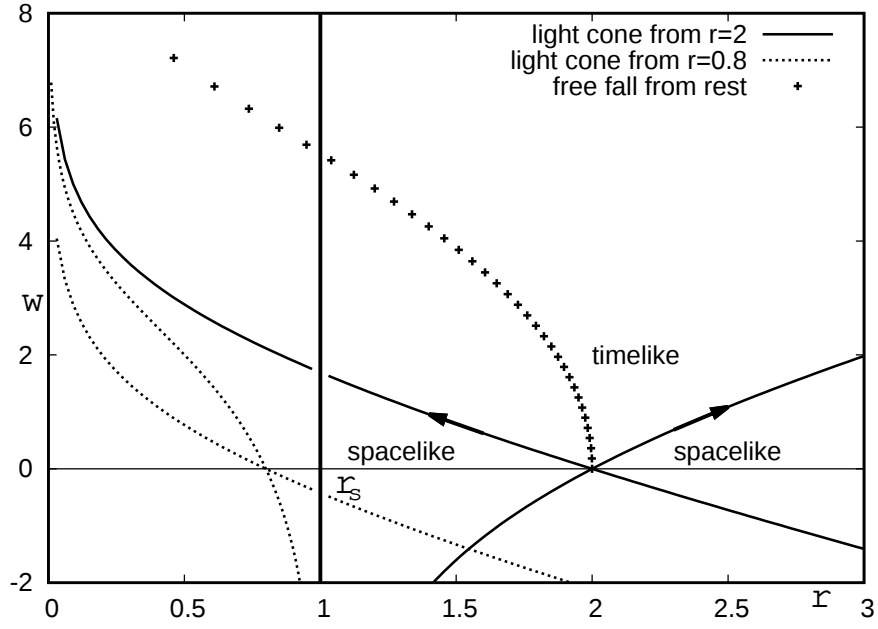


Figure 2: Light cones in the modified coordinates w, r in the vicinity of a black hole. As before, the Schwarzschild radius is taken as unity. One cone shown starts outside the hole at radius 2. The second starts at radius 0.8, inside the black hole. Note how the incoming light wave smoothly crosses the horizon but never reaches the origin. As in Fig. 1, the path of a freely falling object released at $r = 2$ is also shown.

3 Penrose coordinates

Penrose [5] suggested modifying coordinates as an aide to visualizing black hole geometry. Working in polar coordinates to eliminate angular degrees of freedom, it is possible to distort the coordinates so that radial light rays always run along parallel straight lines, with inward and outward null curves remaining perpendicular. Once such coordinates are established, distances are rescaled to map space and time into a finite range.

The resulting diagram is not unique, however such a construction is particularly simple using the coordinates $\{w, r\}$ of the previous section. Starting with any given point in this plane, construct in-going and out-going light-like curves. All such curves can be obtained by translating in w the lines in Fig. 2, and all cross the $w = 0$ axis somewhere. Refer to the crossing points as r_R , and r_L for the outgoing and ingoing light-like curves, respectively. (Inside the horizon the “out-going” light actually moves inward, but more slowly than the “in-going” one.) The mapping between $\{w, r\}$ and $\{r_L, r_R\}$ is one to one, and we can use the latter as intermediate coordinates to describe the full space-time. Positive (negative) w corresponds to $r_L > r_R$ ($r_L < r_R$).

In these coordinates out-going light waves follow constant r_R and in-movers have constant r_L . Now perform a scaling to map infinity to a finite value. For this define

$$v_{L,R} = \arctan(r_{L,R}). \quad (6)$$

Since the coordinates $\{r_L, r_R\}$ each range from 0 to ∞ , the new variables satisfy $0 < v_{L,R} < \pi/2$. Finally, it is conventional to rotate the resulting diagram by $\pi/4$ by defining

$$v_{\pm} = v_R \pm v_L \quad (7)$$

where v_- now represents time and v_+ space. This gives the resulting picture in Fig. 3.

Now in-going light rays are at an angle of $3\pi/4$ and always cross the horizon. Out-going waves are at $\pi/4$ and never touch the horizon. the singularity at $r = 0$ is mapped onto the top of the diagram, which any in-going falling object asymptotically approaches.

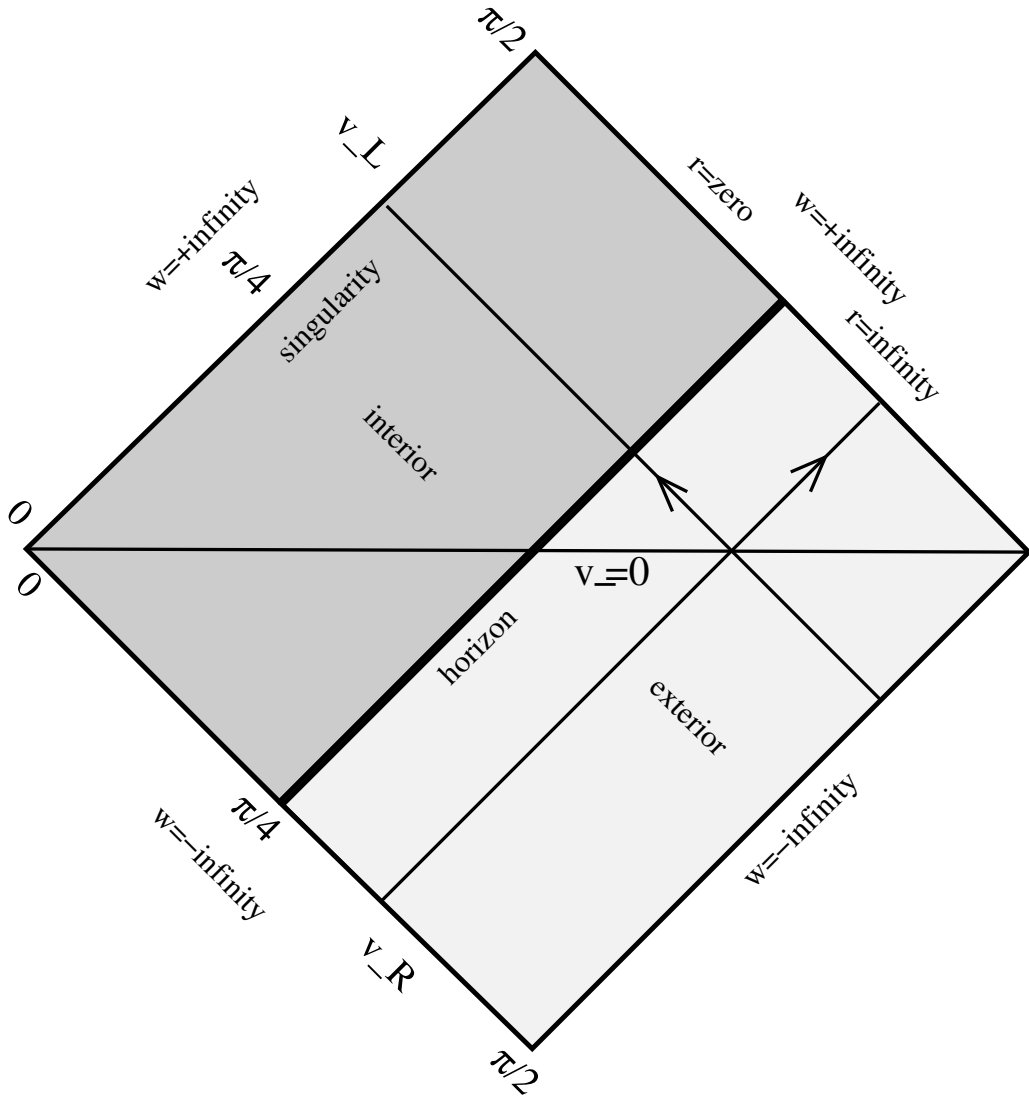


Figure 3: Redrawing the area around the black hole in the new coordinates $v_{\pm}$. In these coordinates the horizon becomes a straight diagonal line. Out-going null trajectories are all parallel to the horizon while in-going ones always cross the horizon.

4 Emerging from a white hole

The metric using w for time is not symmetric under reversal of the sign of w . But reversing this time is effectively a definition of yet another time, τ , defined from the Schwarzschild time by

$$\tau = t - \log(1 - 1/r). \quad (8)$$

for which the metric becomes

$$ds^2 = d\tau^2 (1 - 1/r) + 2dz dr/r^2 - dr^2 (1 + 1/r)(1 + 1/r^2) - r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (9)$$

Again, constant τ surfaces are space-like.

As this is just another choice of coordinates, all physical results must be unchanged from what would see with either the Schwarzschild coordinates or the times w or v_- . We have effectively reflected Fig. 2 vertically about the r axis. Now a freely falling object created inside the black hole can escape to reach points outside. This seems peculiar since we have only changed coordinates, and this cannot not change any observations outside the horizon. The resolution is that the point where this object emerges from the horizon maps into time being minus infinity in either t or w . In essence, this object becomes an initial condition. This peculiar behavior is sketched in Fig. 4.

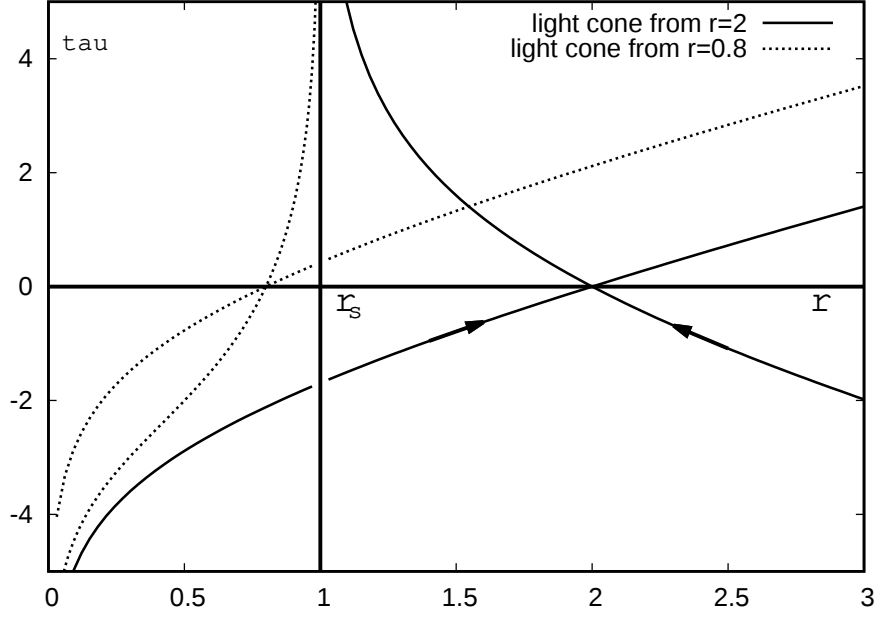


Figure 4: Light cones in the white hole coordinates τ, r . As before, the Schwarzschild radius is taken as unity. One cone shown starts outside the hole at radius 2. The second starts at radius 0.8, inside the black hole. Now out-going light waves do traverse the horizon. In contrast incoming rays always asymptotically approach the horizon, which separates incoming light originating outside the hole from that on the inside. As with the other coordinate choices, two way communication between the interior and exterior regions is forbidden.

Summary

Whichever time one selects, t , w , v_- or τ , is physically arbitrary for an external observer. All choices have space-like constant time surfaces. Two way communication is always forbidden between the interior and exterior of the hole.

References

- [1] Einstein, Albert (1945). "The meaning of relativity," Princeton University Press, ISBN 9780691164083.
- [2] Schwarzschild, K. (1916). "ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". Sitzungsberichte der Kniglich Preussischen Akademie der Wissenschaften. 7: 189196.
- [3] G. Lemaitre (1933). "L'Univers en expansion." Annales de la Socit Scientifique de Bruxelles. A53: 5185.
- [4] Finkelstein, David (1958). "Past-Future Asymmetry of the Gravitational Field of a Point Particle." Phys. Rev. 110: 965967.
- [5] R. Penrose, Gen. Rel. Grav. **43**, 901 (2011). doi:10.1007/s10714-010-1110-5