Phase Transitions

Michael Creutz
Physics Department
Brookhaven National Laboratory
Upton, NY 11973

creutz@bnl.gov

This is a set of notes on phase transitions and critical phenomena prepared to accompany my lectures for the RHIC '97 summer school, held at Brookhaven from July 6 to 16, 1997.

I have been asked to provide an elementary introduction to phase transitions and critical phenomena. The subject is vast; thus, this can really only be a somewhat superficial personal overview of the subject. Many important topics are left out; at the end of these notes is a brief bibliography of sources for further reading.

While most of the universe is a dilute gas of hydrogen, phase transitions are crucial to our existence. We breath air, blood flows through our veins, and our bones are solid. Water boils and lakes freeze over. Mathematically, however, phase transitions are rather remarkable. Statistical mechanics is based on Boltzmann weights

$$P(S) = \frac{e^{-\beta E(S)}}{Z}$$

where the inverse temperature $\beta = \frac{1}{kT}$. From these the partition function

$$Z = \sum_S e^{-\beta E(S)}$$

is constructed. But $e^{-\beta E}$ is an analytic function of β, *i.e.* it has no singularities.

Phase transitions can only occur when an infinite number of states are available, such as with infinite volume V. Start with the partition function

$$Z = \sum_E N(E) e^{-\beta E}$$

where $N(E)$ is the number of states of a given energy E. Each piece of a large system can have its own energy, so one should expect $\langle E \rangle$ proportional to V. Work with the energy density $\mathcal{E} = E/V$. Since there are of order V places to put each bit of energy, we also expect the number of states of a given energy density to grow exponentially

$$N(E) = \exp(V S(\mathcal{E}))$$

This defines the entropy density S at the given energy density. Pulling out the volume factors explicitly,

$$Z = V \int d\mathcal{E} e^{V(S(\mathcal{E})-\beta\mathcal{E})}$$

As $V \to \infty$ the integral is dominated by the maximum of the integrand, where

$$0 = \frac{d}{d\mathcal{E}}(S(\mathcal{E}) - \beta\mathcal{E}) = \frac{\partial S}{\partial E} - \beta$$

or the more usual form

$$\Delta S = \Delta Q/T$$

In this saddle point approximation

$$Z \sim e^{-\beta V F}$$

where free energy is

$$F = \mathcal{E} - S/\beta$$

and all of thermodynamics follows.

A singularity at a phase transition requires structure in $N(E)$. Such structure also requires spatial correlations. Otherwise

$$Z(V) \sim Z(V/N)^N$$

and the free energy becomes just that for a small volume.

In much of the following I will use the Ising model as an example. This has "spins" $s_i \in \{1, -1\}$ occupying lattice sites and has energy $E = -\sum_{\{ij\}} s_i s_j$, where $\{ij\}$ denotes nearest neighbors. Table 1 gives $N(E)$ for the two dimensional Ising model on 10 by 10 lattice with periodic x boundaries, cold y walls. The length of the line gives the entropy. The phase transition is hidden in a subtle flatness.

The numbers in this table are rather large. They add to $2^{100} = 1.267 \times 10^{30}$. Our universe is only about 10^{27} nanoseconds old, suggesting that it is impossible to calculate this list by simple counting. This is a frequent argument for Monte Carlo. See my state-counting papers to learn how I got this list.

How can we get enough correlation for a phase transition? One way is surface tension. Let small volumes be in two possible phases, *i.e.* water and steam. Suppose we pay a penalty for an interface between the phases. For a model, put the system on a lattice with two states in each cell, $s_i \in \{1, -1\}$, representing water and steam

$$Z = \sum_{s_i} e^{-\beta \sum_i F(s_i) + \sum_{nn} J s_i s_j}.$$

Here J represents the surface tension and nn means nearest neighbors. The relative free energies of the two states can be influenced by, say, the pressure. The interesting case is when they are near each other, so let me expand

$$F(s_i) = \overline{F} - H s_i$$

Table 1. State counts for the Ising model on a 10 by 10 lattice.

$E/2+100$	$N(E)$	$E/2+100$	$N(E)$
		100	10880423242637608768349609781 5
0	1	102	127615138775266749696010320050
2	0	104	138682226083589753382353631155
4	100	106	139467535997338317070747513220
6	190	108	129673265537564086898449474485
8	5390	110	111398087687361442602934363604
10	19920	112	88394194656000609637107306835
12	226185	114	64789735278060885125545778420
14	1123330	116	43882526876091406802688842040
16	8441545	118	27484620182084875609413209920
18	46439270	120	15934677821408488316923097025
20	288232165	122	8562769731912107647661352420
22	1596503840	124	4271377195758556988860024315
24	9008597170	126	1981325557749996784540426400
26	48530806690	128	856247175668720270761391365
28	258919598835	130	345440085480687517414714344
30	1348085135068	132	130373941135805243213306725
32	6918375532625	134	46131131663242989983156880
34	34921952998720	136	15336949736067657882440975
36	173864285141465	138	4801625511818556981759340
38	853528946161100	140	1418746354667950902604900
40	4131702217991006	142	396504230728933768862650
42	19698116107747500	144	105044804469611713155910
44	92337394182797240	146	26439076355718752657610
46	424635096183933970	148	6336377505749490029695
48	1910993686546702565	150	1449347253869825330984
50	8394325581182421100	152	317184792213120157975
52	35900636024138056610	154	66590745159525686410
54	149134699701274540190	156	13450173814318534170
56	600434187444808042305	158	2621824707749641960
58	2338237484656296289710	160	494837291835094171
60	8790991827530812668453	162	90726699739843320
62	31852806882802872810510	164	16209249292505960
64	111039862678342970767760	166	2829255985524290
66	371793726574328382611580	168	483344637121035
68	1193670523583033542771745	170	80889449574800
70	3668437423804485582262430	172	13259776474415
72	10772807184138254585743365	174	2126884521530
74	30174747119602748554894980	176	333319272600
76	80467250627920555722255415	178	50912615760
78	203904785227407787528278180	180	7565408818
80	490026517327332203099130689	182	1088231770
82	1114622254786255520262613920	184	151489010
84	2394787743912498152267010800	186	20119550
86	4849969799910449080522379200	188	2579540
88	9239228193366464342451697155	190	303762
90	16521328755364544210468233924	192	35230
92	27673114057688890670065067455	194	3340
94	43328960149817735987320787580	196	350
96	63289600282274727602148469440	198	20
98	86076254527328476831763676120	200	2

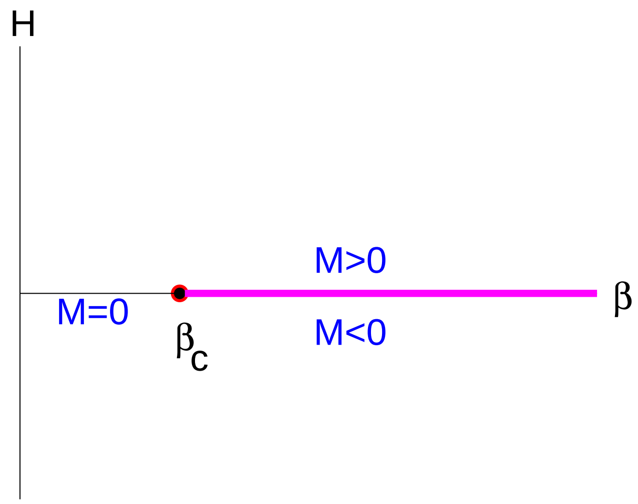

Figure 1. Phase diagram for the Ising model.

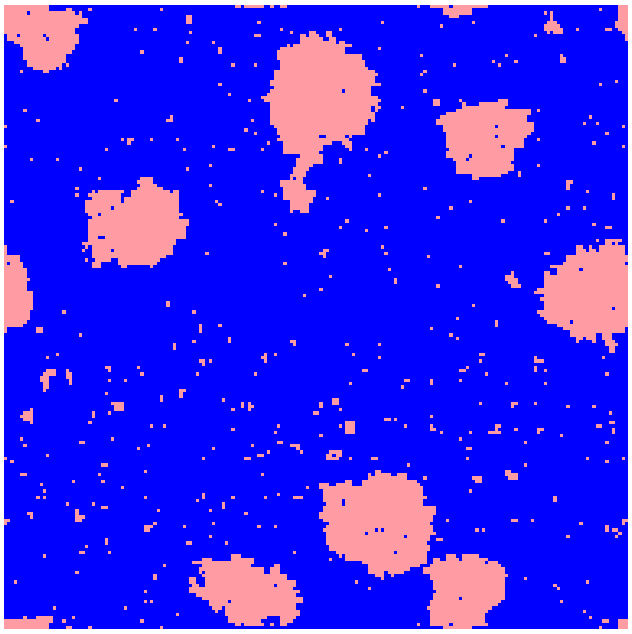
Figure 2. Simulation of boiling water.

where $\overline{F} = (F(1) + F(-1))/2$ and $H = -(F(1) - F(-1))/2$.

Thus we expect the boiling of water to be described by something like the Ising model in an applied field, more usually used for magnets

$$Z = \sum_{s_i} e^{\beta \sum_{nn} s_i s_j + H \sum_i s_i}$$

Boiling occurs on a flip in the sign of H at low temperature. The basic structure of this model, shown in Fig. 1, is a first order line running from a critical β to infinity along the β axis. The $H \leftrightarrow -H$ symmetry, expected for magnets, is broken in water by higher order effects.

Fig. 2 shows a simulation of water boiling, obtained by taking an ordered Ising system and turning on a field in the opposite direction to the magnetization. The bubbles nucleate from larger fluctuations. The picture was made using xpotts, a Potts model simulator from my xtoys collection at http://penguin.phy.bnl.gov/www/xtoys/xtoys.html.

OK, so we need infinite volume, but how infinite? Try a chain of sites in a ring, let the ring length go to infinity. A bond in the chain connecting s and s'contributes

$$T_{s,s'} = e^{\beta s s' + H(s+s')/2} = \begin{pmatrix} e^{\beta+H} & e^{-\beta} \\ e^{-\beta} & e^{\beta-H} \end{pmatrix}_{s,s'}$$

This is the "transfer matrix." Summing over spins gives the partition function for an N site lattice

$$Z = \mathrm{Tr}(T^N)$$

This can be calculated by diagonalizing T

$$Z = \lambda_+^N + \lambda_-^N$$

with

$$\lambda_\pm = e^\beta \left(\cosh(H) \pm \sqrt{\sinh^2(H) + e^{-4\beta}} \right)$$

The free energy per site is dominated by the largest eigenvalue

$$-\beta F = \frac{\log Z}{N} = \log(\lambda_+) + \exp(-N \log(\lambda_+/\lambda_-)) + \ldots$$

Since λ_+ is analytic and positive, the infinite volume free energy has no singularities. We fail to find a phase transition. Note that the finite volume corrections are exponentially small. This shows that the theory has a mass gap. As β goes to infinity with $H = 0$, the eigenvalues become equal and the mass gap goes to zero. The phase transition, to the extent there is one, occurs at zero temperature.

Physically, a kink anti-kink pair has a finite probability, but an infinite number of possible separations. This infinity always disorders the system. In more dimensions a big bubble pays a penalty proportional to its surface, so it is suppressed. For now I will continue to concentrate on the $H = 0$ Ising model, but with more neighbors.

So we need more neighbors. Now for an amusing model. Suppose each site has $2d$ neighbors, but on a system without closed loops. Let me build this lattice recursively. A single outermost site is connected to a deeper neighbor, which has 2d-1 other neighbors. This site can have either spin -1 or spin 1. Fix this spin and define $Z(s)$ to be the partition function obtained by summing over all deeper spins. This partial tree is then recursively built up, defining a Cayley tree as sketched in Fig. 3. To get things started, apply an infinitesimal field to the zeroth level sites

$$Z_0(s) = e^{\epsilon s}$$

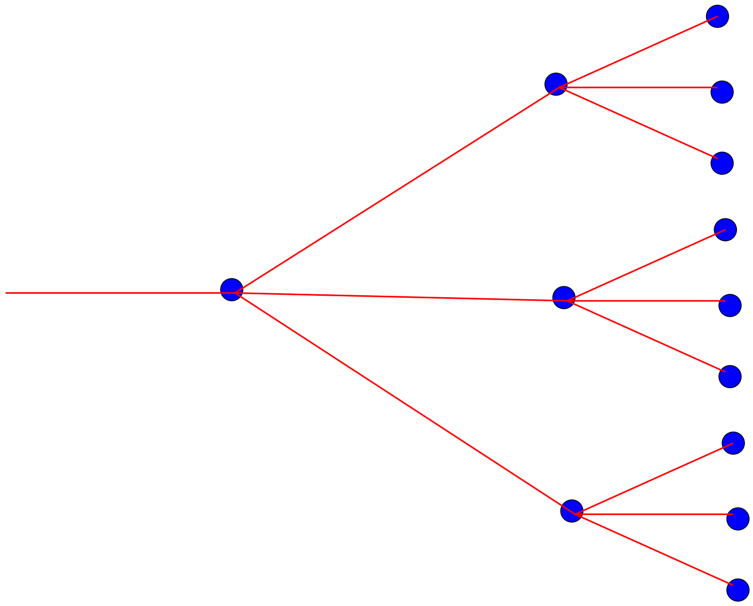

Figure 3. Constructing a Cayley tree.

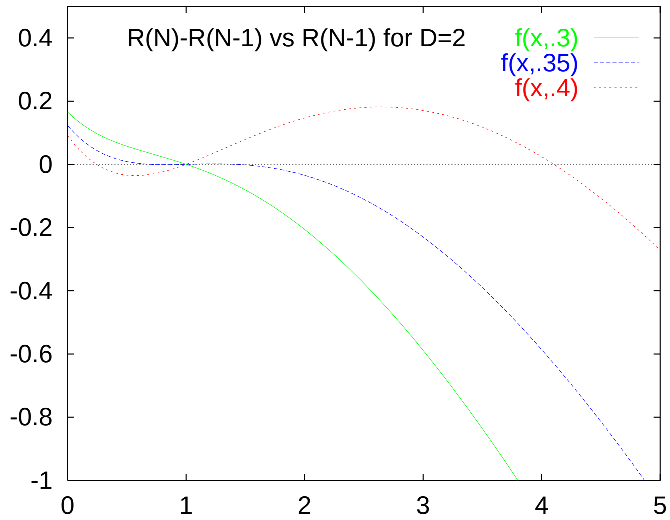

Figure 4. Recursive solution of the Ising model on a Cayley tree.

At level n, the partition function can be written as a sum over the values of the level $n-1$ neighbors

$$Z_n(1) = \left(e^{\beta} Z_{n-1}(1) + e^{-\beta} Z_{n-1}(-1)\right)^{2d-1}$$
$$Z_n(-1) = \left(e^{\beta} Z_{n-1}(-1) + e^{-\beta} Z_{n-1}(1)\right)^{2d-1}$$

Dividing we obtain

$$R_n \equiv \frac{Z_n(1)}{Z_n(-1)} = \left(\frac{e^{\beta} R_{n-1} + e^{-\beta}}{e^{\beta} + e^{-\beta} R_{n-1}}\right)^{2d-1}$$

We can now look for a stable asymptotic solution for a nontrivial R by asking that $R_n = R_{n-1} = R$. As shown in Fig. 4, $R = 1$ is always one solution, but more appear at

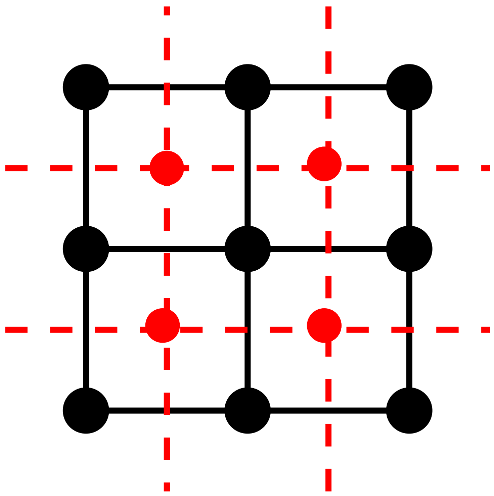

Figure 5. A two dimensional lattice and its dual.

the critical coupling which occurs when the derivative of the right hand side is unity. This happens at

$$\tanh(\beta_c) = \frac{1}{2d-1}$$

When β exceeds the critical value, a value of R above one will flow towards the non-trivial fixed point as seen in the figure. The transition for this model is second order since the fixed point smoothly moves to unity as the critical coupling is approached. This concept of iteration generating a "flow" towards a "fixed point" will be a recurring theme in the following. Note that as d goes to one the critical temperature moves to infinity. One dimension is a critical case.

I now digress into the topic of duality, which gives the exact critical point for the 2d case. To see how this works, change variables from sites to bonds. This can be done a couple of different ways.

For a given configuration, each bond is either excited or not. An excited bond gives a factor of $e^{-\beta}$. A non-excited bond gives e^{β}. Re-express the sum over sites as a sum over bonds being excited or not. These variables over-determine the spins, and thus are constrained; in going around a loop one must encounter an even number of excited bonds. The product of bonds around a plaquette is positive. This view of the system forms the basis for the low temperature expansion, *i.e.* at low temperature most bonds not excited.

Now for an alternative view, write $e^{\beta s_i s_j} = \cosh(\beta) + s_i s_j \sinh(\beta)$. On each bond either take one or the other of these two terms. Assign 1 to bonds with the first choice, -1 to the others. Given a particular configuration of such choices, sum out the spins. This will give zero unless there are an even number of bonds coming out of a site which use the $\sinh(\beta)$ term. This expansion on the bonds is the basis of the high temperature expansion, *i.e.* small β means small $\sinh(\beta)$. Again we get a sum over two states for each bond, with a constraint on these choices. The bond variables multiplied over the four links coming from any bond must be positive.

These two constrained systems are closely related. Transfer the variables from the bonds to dual ones, new bonds crossing the middle of each old one, new sites in the middle

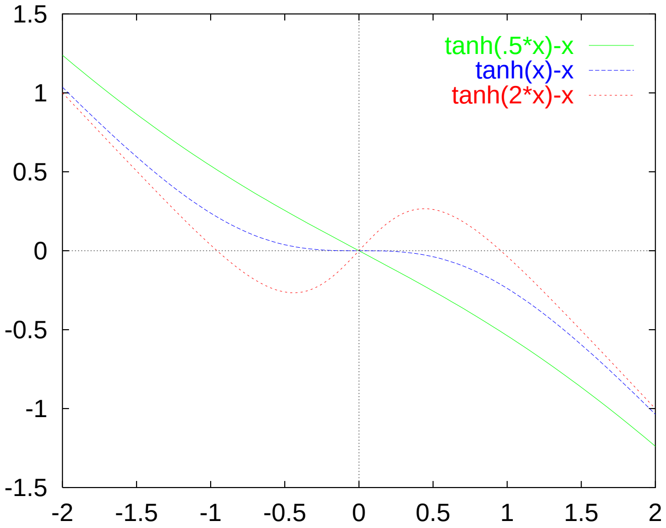

Figure 6. Recursively solving the mean field equation.

of the old plaquettes, as shown in Fig. 5. The product of old bonds out of a site becomes a product of new bonds about the new plaquette. With this mapping on one of the bond descriptions, the two constrained systems now satisfy the same constraint. This means that up to irrelevant factors, the physics at two values of β is related

$$e^{-2\beta} = \tanh(\beta')$$

Under this mapping strong and weak limits interchange. If there is only one singularity, it must occur at the self dual point $\beta = \beta' = \beta_c = \frac{1}{2}\log(1+\sqrt{2}) = .44068\ldots$. This idea of duality extends to Z_n clock models, Potts models, 4-d Z_2 lattice gauge theory, 3d gauge-Higgs system, QED with monopoles, ... Seiberg .... But it all starts here.

I now turn to mean field theory. Consider a large number of dimensions. Then each site has lots of neighbors, suggesting we might assume their effect can be averaged. Suppose we are in a magnetized state with $\langle s\rangle = M$. Look at one spin in the average field of the others. Calculate its magnetization

$$M = \frac{e^{2dM\beta} - e^{-2dM\beta}}{e^{2dM\beta} + e^{-2dM\beta}} = \tanh(2dM\beta)$$

This can be solved by iterating

$$\begin{aligned} &M \to M + \Delta M \\ &\Delta M = \tanh(2dM\beta) - M \end{aligned}$$

We have a non-trivial fixed point only if

$$\beta \geq \beta_c = \frac{1}{2d}$$

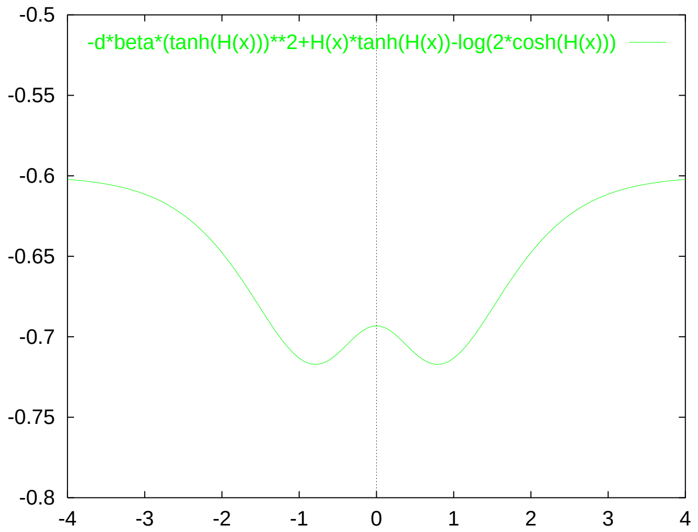

Figure 7. The effective potential from Jensen's inequality.

As shown in Fig. 6, this "flow" like picture gives a similar result to the Cayley tree case at large d, but the approximation misses the transition moving to infinity as the dimension goes to one.

Jensen's inequality provides a more formal approach to mean field theory. Note that e^x is a convex function, $\frac{d^2}{dx^2}e^x = e^x \geq 0$. If x is some stochastic variable, this means that $\langle e^x \rangle > e^{\langle x \rangle}$. Show this for homework.

Consider a "fake" weighting with $P(s_i) = \frac{e^{Hs_i}}{e^H + e^{-H}}$. With this probability distribution

$$\langle s_i \rangle_P = \tanh(H)$$

Thus H might be thought of as a "source" pulling on the spins. Now I manipulate the partition function

$$\begin{aligned}
Z &= \sum_{\{s\}} e^{\beta \sum_{\{ij\}} s_i s_j} \\
&= \sum_{\{s\}} e^{\beta \sum_{\{ij\}} s_i s_j - \sum_i \log(P(s_i))} \prod_i P(s_i) \\
&= \langle e^{\beta \sum_{\{ij\}} s_i s_j - H \sum_i s_i + V \log(2\cosh(H))} \rangle_P \\
&\geq \exp(\langle \beta \sum_{\{ij\}} s_i s_j - H \sum_i s_i + V \log(2\cosh(H)) \rangle_P) \\
&= \exp(V(d\beta \tanh^2(H) - H\tanh(H) + \log(2\cosh(H))))
\end{aligned}$$

Thus for any H the true free energy is less than

$$F <= F_{mf} = -d\beta \tanh^2(H) + H\tanh(H) - \log(2\cosh(H)).$$

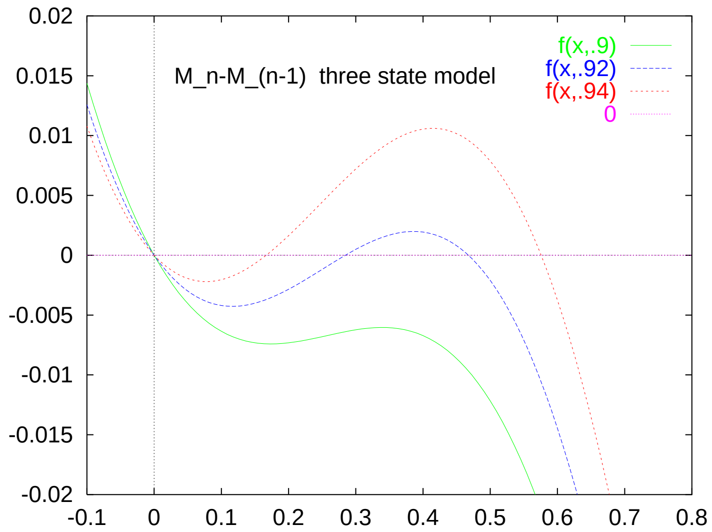

Figure 8. Mean field theory for the three state Potts model.

This is an "effective potential" which, depending on the value of β, can have one minimum at $H = 0$ or a double well shape with two minima, as sketched in Fig. 7. The latter represents the ordered phase. The critical point occurs when $O(H^2)$ term vanishes, *i.e.* the "mass term." This happens at

$$d\beta_c - 1/2 = 0$$

or $\beta_c = \frac{1}{2d}$, as before.

These transitions have been second order, *i.e.* the system evolves continuously with the coupling parameters. The picture is a bit different with three states, where cubic terms can drive us to first order transitions, and physics becomes discontinuous. For example consider the three state Potts model, a system with a Z_3 symmetry. Take $s_i \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$. Suppose the bonds have low energy if the spins are "parallel" or equal, higher energy otherwise

$$E = -\sum_{ij} \mathrm{Re} s_i^* s_j$$

Define the magnetization to vanish for a random distribution

$$M = \langle \mathrm{Re} s_i \rangle$$

For mean field theory, we are to replace neighbors with the average and solve self consistently. Now there are two anti-parallel cases, each giving minus half a unit of magnetization

$$M = \frac{e^{d\beta M} - e^{-d\beta M/2}}{e^{d\beta M} + 2e^{-d\beta M/2}}$$

Expand the right hand side for small M

$$M = \frac{d}{2}\beta M + \frac{d}{8}\beta^2 M^2 + O(M^3)$$

For the Ising case there was no $O(M^2)$ piece, and the $O(M^3)$ piece was negative. As shown in Fig. 8, now a non-trivial solution appears before reaching $\beta = 2/d$. The new solution appears discontinuously; when it becomes of lower energy, we have a first order transition! Note that when the extra solution first appears, it is meta-stable and not the lowest energy; one can use the Jensen inequality argument to estimate when it drops below the unmagnetized state. Simulation results show that for three and more dimensions the transition is indeed first order. In two dimensions, however, it is second order and the prediction fails. In one dimension there is no transition, just as for the Ising case.

Lattice gauge theory is summarized by the path integral

$$Z = \sum_U \exp(\beta \sum_P \mathrm{ReTr} U_P)$$

with $U_P = U_1 U_2 U_3 U_4$, and the U_i are link variables running around the plaquette in question. The local symmetry $U_{ij} \to g_i U_{ij} g_j$ implies there is no barrier to twirling a local group of links around. Without gauge fixing U cannot have an expectation value for any β. This is Elitzur's theorem. One should play the Jensen game for more rigor, but proceed naively anyway, trying to find a self consistent expectation for a link. Do Z_2 for simplicity, which gives the same result as the above Ising case except for the replacement $M \to M^3$ for the average field

$$M = \tanh(\beta M^3)$$

Now there is no linear term at all on the right hand side. The prediction is for a strong first order transition. Most lattice gauge transitions in fact are first order: Z_{2-4} in four dimensions; all known gauge groups in 5 or more dimensions. However, in 3-d, the Z_2 gauge model is dual to the Ising model; so, the transition is second order. In 2-d, gauge fixing turns a gauge model into a one dimensional spin system, with no transitions for any finite dimensional group.

Later in these lectures I will generalize these arguments to suggest a first order deconfining transitions for pure $SU(3)$ gauge theory at finite temperature and for the chiral transition with three massless quarks. This will involve some mathematical formalism that I postpone.

Now I change the subject a bit and remind you of the formal connection between path integrals and statistical mechanics. This is one reason quantum field theorists are also interested in phase transitions. Let me start with a simple quantum mechanics problem with the Hamiltonian

$$H = p^2/2 + V(x)$$

Here p and q are conjugate variables with $[p, q] = -i$. Look at

$$Z = \mathrm{Tr} e^{-\beta H}$$

i.e. zero space dimensional quantum statistical mechanics. As $\beta \to \infty$ we project out the ground state and get ordinary quantum mechanics at zero temperature. This is also the trace of the evolution operator e^{-itH} for imaginary time $t = -i\beta$. Quantum mechanics in imaginary periodic time is the same problem as quantum statistical mechanics.

I now break up β into a large number of "imaginary time slices"

$$Z = \mathrm{Tr} \prod_1^N e^{-\beta H/N}$$

Insert a complete set of states at each slice

$$Z = \int dx_1 \ldots dx_N \prod_{i=1}^N \langle x_i | e^{-\beta H/N} | x_i + 1\rangle$$

where $x_{N+1} \equiv x_1$. Now for large N we can approximate

$$\langle x_i | e^{-\beta H/N} | x_{i+1}\rangle \sim e^{-\beta V(x_i)/N} \langle x_i | e^{-\beta p^2/(2N)} | x_{i+1}\rangle$$

The second factor can be worked out by inserting a complete set of momentum eigenstates

$$\langle x_i | e^{-\beta p^2/(2N)} | x_{i+1}\rangle = \int dp e^{-ip(x_i - x_{i+1})} e^{-\beta p^2/(2N)} = \sqrt{\frac{2}{\pi}} e^{-N(x_{i+1}-x_i)^2/(2\beta)}$$

Thus we have the simple form

$$Z = \int dx_1 \ldots dx_N e^{-S}$$

where the "lattice action" is simply

$$S = a \sum_i V(x_i) + \left(\frac{x_{i+1} - x_i}{a}\right)^2$$

and the lattice spacing $a = \beta/N$. This defines the path integral, which formally is a classical statistical mechanics problem in one dimension. It represents the thermal dynamics of a polymer, with the x_i denoting the coordinates of the atoms in the chain.

Going from a single particle to a field theory, D space dimensional quantum mechanics is equivalent to $d = D + 1$ dimensional classical statistical mechanics. The infinite "time" limit gives the ground state, while finite imaginary time relates quantum statistical mechanics to classical statistical mechanics in one more dimension.

Second order phase transitions are essential to continuum limits. Taking the lattice spacing to zero requires physical correlation lengths to diverge in lattice units. The particle physicist's e^{-mr} corresponds to the statistical mechanic's $e^{-n/\xi}$. With lattice spacing a, we identify $r = na$ and $\xi = m/a$. A continuum limit requires $\xi \to \infty$, which occurs at a phase transition, most particularly at a second order one. For lattice gauge theory, 4-d is a borderline case and the transition occurs at $\beta \sim 1/g^2 = \infty$, with the approach given by standard asymptotic freedom arguments.

Now for another jump in subject. Despite the usual pedagogical approaches, effective potentials want to be convex. In field theory language, consider

$$Z = \int d\phi e^{-S(\phi)}$$

Adding in some external sources

$$Z(J) = \int d\phi e^{-S(\phi)+J\phi}$$

general correlation functions can be found by differentiating with respect to J. Here I shorthand notate $J\phi = \int dx J(x)\phi(x)$ in the continuum, or $J\phi = \sum_i J_i \phi_i$ on the lattice. Think of J as an external force pulling on the field. Such a force will give the field an expectation value

$$\langle \phi \rangle_J = -\frac{\partial F}{\partial J}$$

where I define the free energy $F(J) = -\log(Z(J))$. Now imagine inverting this to find what force $J(\Phi)$ gives some desired expectation value, *i.e.* solve

$$\Phi(J) = \langle \phi \rangle_{J(\Phi)} = -\frac{\partial F}{\partial J}$$

In terms of this formal solution, construct the "Legendre transform"

$$V(\Phi) = F(J(\Phi)) + \Phi J(\Phi)$$

and look at

$$\frac{\partial V}{\partial \Phi} = -\Phi \frac{\partial J}{\partial \Phi} + J + \Phi \frac{\partial J}{\partial \Phi} = J$$

If I turn off the sources, this derivative vanishes. Thus the minimum of V tells us the expectation value of the field. This quantity V is the "effective potential."

But now let me confuse you by looking at the second derivative of V

$$\frac{\partial^2 V}{\partial \Phi^2} = \frac{\partial J}{\partial \Phi}$$

Actually, it is easier to look at the inverse

$$\frac{\partial \Phi}{\partial J} = -\frac{\partial^2 F}{\partial J^2} = \langle \phi^2 \rangle - \langle \phi \rangle^2 = \langle (\phi - \langle \phi \rangle)^2 \rangle \geq 0.$$

Thus this second derivative has a single sign! This shows we are actually looking for a minimum and not a maximum of V, but in addition it implies that V can only have ONE minimum!

So what is going on? Are phase transitions impossible? The more you pull, the larger the expectation of Φ should be. It won't go back. Physically, we must do Maxwell's construction. If we force the expectation of ϕ to lie between two distinct stable phases, the system phase separates into a mixture of the two. Note that there is no large volume limit in the above discussion. However other definitions of V can allow a small barrier at finite volume due to surface tension effects. A mixed phase must contain interfaces, and their energy represents a barrier.

First order transitions have a discontinuity in the internal energy, representing a latent heat. The barrier in the effective potential (modulo the above discussion) allows meta-stability and hysteresis. Water in a clean container can "bump" rather unpleasantly. Explosives last in a meta-stable state for long periods. Are things actually analytic as you pass through the transition? No, there is an essential singularity that I will now discuss. The free energies of the phases match at the transition; suppose I can expand as we go through it

$$\Delta F = C(\beta - \beta_c)$$

Let the surface tension between the two phases be σ. Creating a bubble of radius r costs free energy

$$E(r) = \frac{4\pi r^3}{3}\Delta F + 4\pi r^2 \sigma$$

This has maximum energy at

$$0 = 4\pi r^2 \Delta F + 8\pi r\sigma$$

or

$$r = -2\sigma/\Delta F$$

At this point the energy of the bubble is

$$E_{max} = \frac{16\pi\sigma^3}{3(\Delta F)^2}$$

As we approach the transition, this radius goes to infinity and this "semi-classical" argument becomes rigorous. This energy represents a barrier to bubble nucleation, which is suppressed by the Boltzmann weight

$$P \sim \exp(-\beta E_{max}) = \exp(-\frac{C'}{(\beta - \beta_c)^2})$$

An essential singularity appears in the physics as one passes into a meta-stable state. Even though things look very analytic, they are not. Since the meta-stable state can decay, this expression represents an "imaginary part" for the free energy of the unstable phase.

Now for a brief discussion on some aspects of Goldstone Bosons. Suppose I have a conserved current

$$\partial_\mu j_\mu = 0$$

so the corresponding charge $Q = \int d^3x j_0(x)$ is a constant

$$\frac{dQ}{dt} = -i[H, Q] = 0.$$

Suppose, however, that the vacuum is not a singlet under this charge

$$Q|0\rangle \neq 0$$

Then there cannot be a mass gap in the theory. Consider the state

$$\exp(i\theta \int d^3x j_0(x) e^{-\epsilon x^2})|0\rangle$$

where ϵ is a convenient cutoff and θ some parameter. As epsilon goes to zero this state by assumption is not the vacuum, but the expectation value of the Hamiltonian goes to zero (normalize so the ground state energy is zero). Thus "spontaneously broken symmetries" have no mass gap, *i.e.* the theory contains states of arbitrarily low energy. These are manifested as massless particles called Goldstone bosons.

Free massless field theory is a marvelous example of all this where everything can be worked out. The massless equation of motion

$$\partial_\mu \partial_\mu \phi = 0$$

can be written in the form

$$\partial_\mu j_\mu = 0$$

where

$$j_\mu = \partial_\mu \phi.$$

The broken symmetry is the invariance of the Lagrangian $L = \int d^4x (\partial_\mu \phi)^2$ under shifts of the field

$$\phi \to \phi + c$$

Note that $j_0 = \partial_0 \phi = \pi$, the conjugate variable to ϕ. One can work out explicitly

$$\langle 0| \exp(i\theta \int d^3x j_0(x) e^{-\epsilon x^2/2})|0\rangle.$$

but we can save ourselves the work using dimensional analysis. The field ϕ has dimensions of inverse length, while j_0 goes as inverse length squared. Thus θ above has units of inverse length. These are the same dimensions as ϵ^2. Now for a free theory by Wick's theorem the answer must be Gaussian in θ, so we conclude that the above overlap must go as

$$\exp(-C\theta^2/\epsilon^4)$$

where C is some non-vanishing dimensionless number. This expression rapidly goes to zero as epsilon becomes small, showing that the vacuum is indeed not invariant under the symmetry. As ϵ goes to zero, we obtain a new vacuum that is not even in the same Hilbert space. Its overlap with any polynomial of fields on the original vacuum vanishes.

Two dimensions give some interesting twists on this argument. Now the scalar field is dimensionless, and the current has dimensions of inverse length. Thus theta is dimensionless and we expect

$$\langle 0|\exp(i\theta\int dx j_0(x) e^{-\epsilon x^2/2})|0\rangle \sim \exp(-C\theta^2)$$

There can be no $\log(\epsilon)$ since there is nothing to set the scale. Thus the vacuum is not invariant, but the symmetry relation does not give you a fully independent state. This is clearly a borderline case, and for an interacting theory the massless particles can be lost.

This is tied in with the propagator in two dimensions not being a distribution. Put in a small mass cutoff. Then

$$\Delta(x) = \int \frac{d^2p}{(2\pi)^2}\frac{e^{-ipx}}{p^2+m^2}$$

gets very singular as m goes to zero. Consider the simple test function $e^{-x^2/2}$. This is infinitely differentiable and well behaved at infinity. Now integrate this test function with the free particle propagator

$$\int d^2x \Delta(x) e^{-x^2/2} \sim \int \frac{d^2p}{(2\pi)^2}\frac{e^{-p^2/2}}{p^2+m^2} \sim \log(1/m) \Longrightarrow_{m\to 0} \infty$$

Thus $\Delta(x)$ is not a tempered distribution, contrary to the basic assumptions of quantum field theory. However Green's functions of j_0 are tempered distributions since they involve derivatives that kill the divergent part. In most cases Goldstone bosons are lost in two dimensions, however, if they are free, as in the above case, they can exist. The X-Y model, with spins in $U(1)$, has a massless phase, but no long range order. Lore is that higher symmetries only have massive phases, but Seiler and Patrascioiu have argued that this may be wrong.

I now turn to the renormalization group, which I approach via the Migdal-Kadanoff approximate recursion relations. I start with a discussion of decimation. Let us go back to the Ising model in one dimension

$$Z = \mathrm{Tr}\begin{pmatrix} e^{\beta+H} & e^{-\beta} \\ e^{-\beta} & e^{\beta-H}\end{pmatrix}^N = \mathrm{Tr} T^N$$

Let me sum over every other spin, giving

$$Z = \mathrm{Tr}(T')^{N/2}$$

where

$$T' = T^2 = \begin{pmatrix} e^{2(\beta+H)} + e^{-2\beta} & e^H + e^{-H} \\ e^H + e^{-H} & e^{2(\beta-H)} + e^{-2\beta}\end{pmatrix}$$

We now match this with the original form of T

$$T' = C \begin{pmatrix} e^{\beta'+H'} & e^{-\beta'} \\ e^{-\beta'} & e^{\beta'-H'} \end{pmatrix}$$

We see that exactly the same physics occurs on a lattice of twice the spacing and new couplings (β', H'). The values of C, β' and H' are fixed by the three equations

$$\begin{aligned} Ce^{-\beta'} &= e^H + e^{-H} \\ Ce^{\beta'+H'} &= e^{2(\beta+H)} + e^{-2\beta} \\ Ce^{\beta'-H'} &= e^{2(\beta-H)} + e^{-2\beta} \end{aligned}$$

This process is called decimation, *i.e.* integrating out some of the degrees of freedom. To simplify the equations, I turn off H, obtaining

$$\beta' = \frac{1}{2}\log(\cosh(2\beta))$$

This can be written in a form reminiscent of our earlier recursions

$$\beta' - \beta = -\frac{1}{2}\log\left(\frac{2}{1+e^{-4\beta}}\right)$$

The only fixed point occurs at $\beta = 0$. The new coupling is always less than the old one as long as beta is positive. Repeating this as an iteration drives any β to zero.

It is instructive to extend this to non integer decimations. For this, write the transfer matrix in the form $T_{ss'} = \cosh(\beta)(1 + ss'\tanh(\beta))$. The above decimation by a factor of two involves the sum

$$\tfrac{1}{2}\sum_{s_2}(1+s_1s_2t)(1+s_2s_3t) = (1+s_1s_3t^2)$$

or simply $\tanh(\beta) \to \tanh^2(\beta)$. Interpolate this to rescaling by a factor of $1+\Delta$, taking $\tanh(\beta) \to \tanh^{1+\Delta}(\beta)$. Infinitesimally, this reduces to

$$\frac{\beta'-\beta}{\Delta} \sim a\frac{d\beta}{da} = \cosh(\beta)\sinh(\beta)\log(\tanh(\beta))$$

This is the renormalization group equation for this system. The right hand side is negative for all positive β. As the lattice spacing varies from zero to infinity, the coupling β flows from the ultraviolet fixed point at infinity to the infrared fixed point at zero.

Suppose a system has a non-trivial fixed point satisfying

$$a\frac{d\beta}{da} = \lambda(\beta - \beta_c).$$

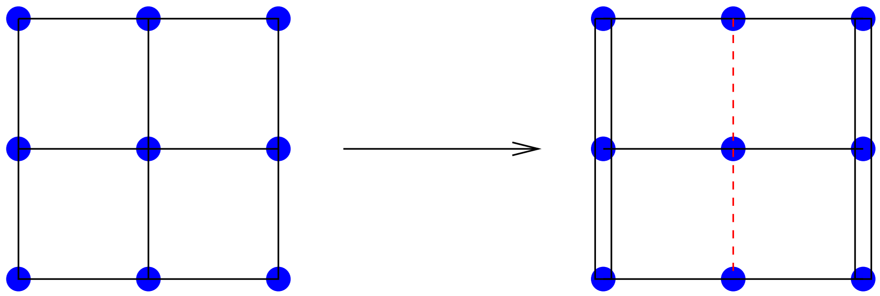

Figure 9. Moving bonds around.

This has the solution

$$\beta = \beta_c + Ca^{\lambda}$$

Since $a \sim 1/\xi$, this says

$$1/\xi \sim (\beta - \beta_c)^{1/\lambda}$$

This is the renormalization group way of seeing how non-trivial exponents can arise as one approaches a critical point.

Going on to more dimensions we loose the exactness and must make approximations. Integrating out a site in more than one dimension introduces couplings between all sites to which it is coupled. Integrating the sites along a line couples all spins attached to that line. Integrating out all but the corners on a block requires an infinite number of couplings. This makes things less than rigorous, but can imagine a similar coupling "flow" in a higher space.

Making an approximation by moving bonds around allows one to analytically study these flows. Imagine integrating out every other site in say the x direction. To avoid long range couplings being generated in the y direction, follow Kadanoff and "move" the y bonds to sites not being integrated over, as sketched in Fig. 9. Every second y bond becomes twice as strong, and then the earlier x decimation can be carried out on the remaining sites. Thus we relate the model at β_x, β_y to that at

$$\begin{aligned} \beta_x' &= \frac{1}{2}\log(\cosh(2\beta_x)) \\ \beta_y' &= 2\beta_y \end{aligned}$$

Now repeat this for the y direction. The resulting transformation is asymmetric due to the approximations. To get more symmetric, do things differentially, using the earlier equation for the x coupling and $a\frac{d\beta}{da} = \beta$ for the bond moving. The total change of coupling is then

$$a\frac{d\beta}{da} = \cosh(\beta)\sinh(\beta)\log(\tanh(\beta)) + (d-1)\beta$$

Here I insert a factor of $d-1$ to allow for bond moving in all directions but the decimation one. The result is exact in one dimension, and for $d = 2$ it still gives the exact β_c. I plot this function in Fig. 10 for $d = 2$.

The renormalization group relates theories with different lattice spacings. If we could keep track of an infinite number of couplings, the procedure would be "exact," but in

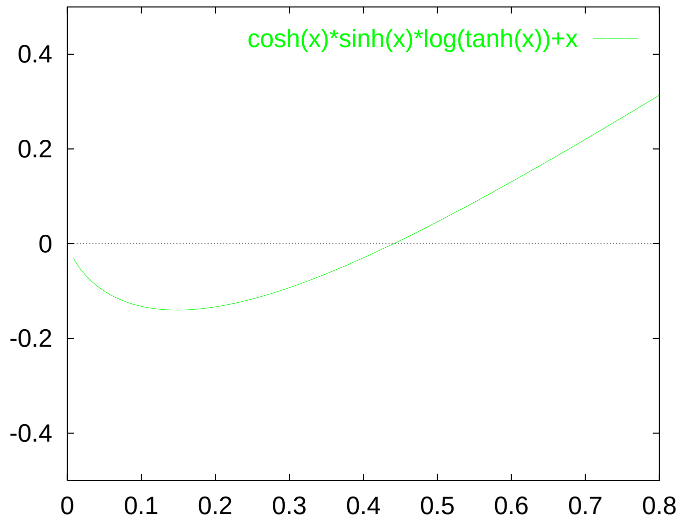

Figure 10. The Migdal-Kadanoff recursion relation for $d = 2$.

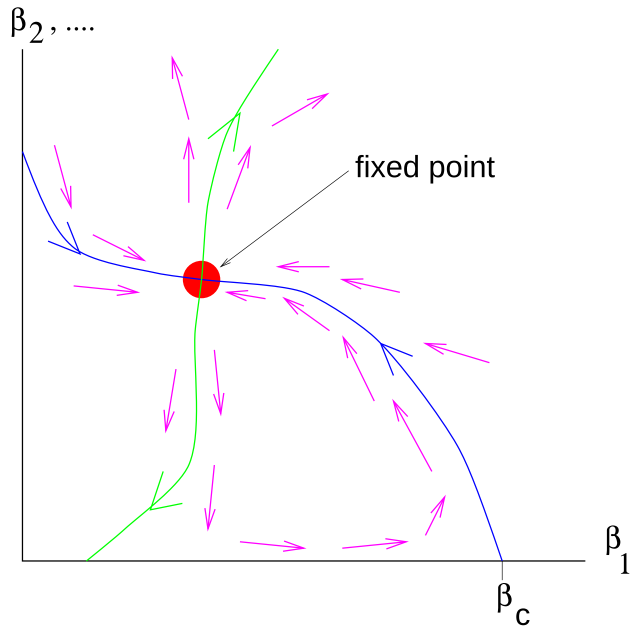

Figure 11. A generic renormalization group flow.

reality we usually need some truncation. Continuing to integrate out degrees of freedom, the couplings flow and might reach some “fixed point.” With two couplings, there can be an attractive “sheet” towards which couplings flow, and then they go towards the fixed point, as sketched in Fig. 11. If the fixed point has only one attractive direction, then two different models that flow towards that same fixed point will have the same physics. This is universality, *i.e.* exponents are the same for all these models with the same attractor.

So mean field theory describes a phase transition in terms of a changing classical ground state as parameters are varied, and the renormalization group description is in terms of a flow through a complex coupling constant space. When should we rely on which picture?

Some hints come from dimensional analysis, although, in ignoring non-perturbative effects that might occur at strong coupling, the following arguments are not rigorous. In d dimensions a conventional scalar field has dimensions of $M^{\frac{d-2}{2}}$. Thus the coupling constant λ in an interaction of form $\int d^dx\ \lambda\phi^n$ has dimensions of $M^{d-n\frac{d-2}{2}}$. On a lattice of spacing a, the natural unit of dimension M is the inverse lattice spacing. Thus without any special tuning, the renormalized coupling at some fixed physical scale would naturally run as $\lambda \sim a^{n\frac{d-2}{2}-d}$. As long as the exponent in this expression is positive, i.e.

$$n \geq \frac{2d}{d-2}$$

we expect the coupling to become "irrelevant" in the continuum limit. The fixed point is driven towards zero in the corresponding direction. If d exceeds four, this is the case for all interactions. (I ignore ϕ^3 in 6 dimensions because of stability problems.) This suggests that four dimensions is a critical case, with mean field theory giving the right qualitative critical behavior for all larger dimensions. In four dimensions we have several possible "renormalizable" couplings which are dimensionless, suggesting logarithmic corrections to the simple dimensional arguments. Indeed, four-dimensional non-abelian gauge theories should display exactly such a logarithmic flow; this is asymptotic freedom.

This simple dimensional argument applied to the mass term suggests it would flow towards infinity in all dimensions. For a conventional phase transition, something must be tuned to a critical point. In statistical mechanics this is the temperature. In field theory language we usually remap this onto a tuning of the mass term, saying that the transition occurs as some scalar mass goes through zero. This tuning of scalar mass terms required for a continuum limit seems unnatural and is one of the unsatisfying features of the standard model, driving particle physicists to try to unravel how the Higg's mechanism really works.

Recently there has been considerable interest in statistical systems that become critical without any tuning of parameters. This phenomenon of "self-organized criticality" may explain the fractal structure of much of the world around us. However field theoretical applications of this concept remain elusive.

In non-Abelian gauge theories with massless fermions, chiral symmetry protects the mass from renormalization, avoiding any special tuning. Indeed, these models exhibit the amazing phenomenon of dimensional transmutation: all dimensionless parameters in the continuum limit are completely determined by the basic structure of the initial Lagrangian, without any continuous parameters to tune. In the limit of vanishing pion mass, the rho to nucleon mass ratio should be determined from first principles; it is the goal of lattice gauge theory to calculate just such numbers.

As we go below four dimensions, this dimensional argument suggests that several couplings can become "relevant," requiring the renormalization group picture of flow towards a non-trivial fixed point. Above two dimensions the finite number of renormalizable couplings corresponds to the renormalization group argument for a finite number of "universality classes," corresponding to different basic symmetries.

One might imagine dimensionality as being a continuously variable parameter. Then just below four dimensions a renormalizable coupling becomes "super-renormalizable" and a new non-trivial fixed point breaks away from vanishing coupling. Near four dimensions this point is at small coupling, forming the basis for an expansion in $4-d$. This has become a major industry, making remarkably accurate predictions for critical exponents in three dimensions.

Now I return to lattice gauge theory and discuss how pure glue at finite temperature mimics a three state model and might be expected to have a first order deconfining transition. I lead into this with a bit of group theory. Consider some compact group G with elements g. There exists a unique measure

$$\int dg f(g) = \int dg f(g_1 g) = \int dg f(g g_1)$$

where I normalize $\int dg = 1$. (Non-compact groups might have different normalizations for left and right.) For example, with $U(1)$ I can take $g = e^{i\theta}$ and then $\int dg = \int_0^{2\pi} d\theta/2\pi$. For $SU(2)$, write $g = a_0 + i\vec{a}\cdot\vec{\sigma}$ and then $\int dg = \int d^4a\ \delta(a_0^2 + \vec{a}^2 - 1)$, *i.e.* the surface of a four dimensional sphere, an S_3.

This group integration extracts the "singlet" part of a function in the following sense. Suppose f is a "class function," *i.e.* $f(g) = f(g_1 g g_1^{-1})$. Then I can expand it in traces over the various irreducible representations R of the group

$$f(g) = \sum_R f_R \chi_R(g)$$

where the character $\chi_R(g) = \mathrm{Tr} M_R(g)$ and $M_R(g)$ is the matrix representing g in representation R. These representations include the trivial one, $R = 0$, the fundamental one $R = F$, the adjoint $R = A$, and generally infinitely many more. For irreducible representations, the characters χ satisfy an orthogonality condition

$$\int dg\ \chi_R^*(g)\ \chi_{R'}(g) = \delta_{RR'}$$

From this and $\chi_0(g) = 1$, we see

$$\int dg f(g) = f_0.$$

If we insert a character in some other representation we obtain

$$\int dg \chi_R^*(g)\ f(g) = f_R.$$

This allows us to do some specific integrals integrals quite easily. For example, with $SU(3)$ we have

$$\int_{SU(3)} dg\ (\mathrm{Tr} g)^3 = 1$$

since there is only one singlet in the famous decomposition $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. This integral lies at the base of the argument below for a first order chiral transition with three massless quark flavors.

So lets apply this to lattice gauge theory. On each bond of our hyper-cubic lattice we have a group element U_{ij}. The Wilson action multiplies these around elementary squares and constructs $U_P = U_1 U_2 U_3 U_4$. The partition function is

$$Z = \int \{dU\} e^{-\beta \sum_P \mathrm{Re}\chi_F(U_P)}$$

Formally, as argued earlier, this represents something like $Z = \mathrm{Tr} e^{aHN_t}$ with a the temporal lattice spacing. The picture, however, is a bit more complicated due to gauge invariance. If we put a group element g_i on each site, we can imagine taking $U_{ij} \to g_i U_{ij} g_j^{-1}$. This change cancels from the action.

Gauge invariance leads to the possibility of gauge fixing. This can be done much more generally, but for now suppose I forget to integrate over one link and define

$$Z(U_0) = \int \{dU\} \ \exp(-\beta \sum_P \chi_0(U_P)) \ \delta(U_{ij}, U_0)$$

On the integrand I can do a gauge transformation and then use the invariance of measure to find

$$Z(U_0) = Z(g_i U_0 g_j^{-1})$$

So $Z(U_0)$ doesn't depend on U_0. Since my measure is normalized, $Z = Z_0$. I can continue this and forget to integrate over more links. As long as no closed loops are fixed, then the partition function is unchanged. A closed loop is a gauge invariant observable, so we better not be able to fix it.

In the temporal gauge, all time-like links are set to unity. In this gauge the above transfer matrix argument reduces the path integral to Hamiltonian lattice gauge theory. But on finite periodic temporal lattice, temporal links at a given spatial site form a closed loop. Thus one cannot gauge fix all of them. At each spatial location we must leave one temporal link unfixed, take it to be at time 0. What does integrating over this link correspond to?

In Hamiltonian language, there is an operator $R_i(g)$ that does a local gauge rotation at site i. In particular, for a link to a spatial neighbor

$$R_i(g) U_{ij} R_i^{-1}(g) = g U_{ij}$$

These are all operators in the Hilbert space of the Hamiltonian approach. What the path integral formally reduces to is

$$Z = \mathrm{Tr}\left(e^{-aHN_t} \prod_i (\int dg_i R_i(g_i)) \right)$$

From the above discussion we see that this last integration projects out a gauge singlet. This operator imposes the lattice-gauge-theory version of Gauss's law

$$\delta(\vec{D} \cdot \vec{E}) \leftrightarrow \int dg R_i(g)$$

Now we can generalize and consider not projecting out the singlet everywhere. In particular, at one site I might want to put down a quark-like source. To do this I simply insert the character for the desired representation

$$\int dg \chi_F^*(g) R_i(g)$$

The ratio of the new partition function to the old is the Wilson line or the Polyakov loop. Going back to the path integral, it is just the expectation of a product of temporal links wrapping around the time direction. This Hamiltonian argument explicitly shows how it represents the energy carried by a fixed source in the fundamental representation of the gauge group.

For the quark-less theory, the unfixed temporal links at time 0 have a global symmetry under the center of the gauge group. For the $SU(3)$ of the strong interactions the center is the set $\{1, e^{\pm 2\pi i/3}\}$. By definition, center elements commute with all group elements, and the global change $g_i \rightarrow e^{2\pi i/3} g_i$ will cancel out of the temporal plaquettes, each of which involves one g_i and one g_j^{-1}. This is exactly the same symmetry as for the 3 state Potts model, which I argued above should have a first order phase transition. This prediction is well verified by numerical simulation.

The quark kinetic term explicitly breaks this symmetry, so the transition might go away. For massless quarks, it is instead the global chiral symmetry that becomes relevant. This suggests second order for two flavors. For three light flavors the suggestion is first order since there is a quadratic term in the mean field expansion. This arises since the product of three fundamental representations contains a singlet piece, as we well know because three quarks can combine to form a gauge singlet baryon. Generalizing our earlier mean field equation to an $SU(3)$ spin system gives

$$3M = \frac{\int dg\ e^{2d\beta M \mathrm{ReTr} g} \mathrm{Tr} g}{\int dg\ e^{2d\beta M \mathrm{ReTr} g}} = \frac{d\beta M + (d\beta M)^2/2 + \ldots}{1 + (d\beta M)^2 \ldots} = d\beta M + (d\beta M)^2/2 \ldots$$

where I use the earlier $SU(3)$ example integral. The quadratic term means that the solution jumps discontinuously, just as argued earlier for the Potts model. The interpolation between the small and large mass limits for various numbers of flavors is a major area of current study.

A few general references

P. Bak and M. Creutz, 1994, "Fractals and self-organized criticality," in *Fractals in science*, Armin Bunde and Shlomo Havlin eds., p. 26 (Springer-Verlag). A review of self-organized criticality.

Coleman, S., 1973, *Commun. Math. Phys.* **31**, 259. A discussion of Goldstone Bosons in two dimensions.

Coleman, S., 1975, in *Laws of hadronic matter*, A. Zichichi, ed., p. 139 (Academic Press). An excellent discussion of effective potentials and spontaneous symmetry breaking.

Coleman, S. and E. Weinberg, 1973, *Phys. Rev.* **D16**, 1888. Introduces dimensional transmutation.

Creutz, M., 1983, *Quarks, gluons and lattices* (Cambridge University Press). An introduction to lattice gauge theory.

Itzykson, C. and J.-M. Drouffe, 1989, *Statistical field theory*, (Cambridge University Press). A nice general discussion of many of the topics in these notes, especially duality, Cayley trees, and mean field theory.

Kadanoff, L. 1976 *Ann. Phys.* **100**, 359; *Rev. Mod. Phys.* **49**, 267; Migdal, A. 1975, *Zh. Eksp. Teor. Fiz.* **69**, 810,1457 [*Sov. Phys. Jetp.* **42**, 413, 743]. The Migdal-Kadanoff recursion relations.

Patrascioiu, A. and E. Seiler, 1997, hep-lat/9706011. One of a series of papers questioning common but unproven lore on two dimensional models.

Shultz, T., D. Mattis, E. Lieb, 1964, *Rev. Mod. Phys.* 36, 856. An elegant solution of the two dimensional Ising model in terms of domain walls being free fermions.

Wilson, K., (1971), *Phys. Rev.* B4, 3174, 3184. The basic renormalization group idea as applied to critical phenomena.