Surface states and chiral symmetry on the lattice

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In a Hamiltonian formalism we study chiral symmetry for lattice fermions formulated in terms of Shockley surface states bound to a wall in an extra spatial dimension. For hadronic physics this provides a natural scheme for taking quark masses to zero without requiring a precise tuning of parameters. We illustrate the chiral anomaly as a flow of states in this extra dimension. We discuss two alternatives for extending the picture to a chiral coupling of gauge fields to such fermions: one with a small explicit breaking of gauge symmetry and one with heavy mirror fermions.

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I. INTRODUCTION

Chiral symmetry has long played an essential role in particle theory. The pion is made of the same quarks as the ρ meson, yet its mass is considerably less. The canonical explanation says that, if quarks were massless, then the pions would be Goldstone bosons arising from the spontaneous breaking of an underlying chiral symmetry. Indeed, pure gauge interactions are helicity conserving, and thus both the number of left- and right-handed massless quarks are separately conserved. Through confinement into the physical states of baryons and mesons, this symmetry is spontaneously broken. A host of predictions from current algebra are based on this picture [1].

These issues are complicated by the presence of "anomalies." Ultraviolet divergences make it impossible, even in perturbation theory, to conserve simultaneously all axial vector currents associated with chiral symmetry, and the vector current of electric charge. As current conservation is crucial to our understanding of gauge symmetries, we must conserve the vector current, implying that the axial symmetry cannot be exact.

One consequence of the anomaly is that there is one less Goldstone boson than naive counting would suggest. For two flavors of light quarks, of the four ways to form pseudoscalar mesons there are three light pions while the η remains heavier. With SU(3) flavor symmetry, it is the η' which is anomalously heavy compared to the other pseudoscalars.

Over the last 20 years, lattice techniques have become the dominant method for the study of nonperturbative quantum field theory. The understanding of chiral symmetry in this framework has remained an infamous and elusive goal for the entire period. For a review, see Ref. [2]. The problems with lattice chiral symmetry are intricately entwined with the chiral anomaly. Indeed, as the lattice is a regulator, anomalies can only arise through nonchirally symmetric terms in the underlying action or Hamiltonian. Simple prescriptions without such terms are plagued with the so called "doubling" problem, wherein extra, usually unwanted, fermionic species appear to cancel any anomalies in the theory.

Chiral issues arise in an even more fundamental way with the weak interactions. Here parity violation seems to maximally differentiate between left- and right-handed fermions. While lattice methods have been dominantly applied to the strong interactions, there are reasons to desire a lattice formulation of the weak interactions as well. In particular, the lattice is the best founded nonperturbative regulator, and thus provides an elegant framework for the definition of a quantum field theory. Even though the smallness of the electromagnetic coupling makes nonperturbative effects quite small in the electroweak theory, at least in principle we would like a rigorous formulation. While exceptionally small, some interesting nonperturbative phenomena are directly related to the anomaly, such as the prediction of 't Hooft [3] that baryons can decay through the instanton mechanism.

The last year has seen considerable theoretical activity on the use of surface states as a basis for a theory of chiral lattice fermions [4]. Here one envisions our four-dimensional world as an interface embedded in a five-dimensional underlying space. With appropriate conditions on the model, low energy fermionic states are bound to this wall. With these being the only low energy states, we have an effective chiral theory on the interface. The prime purpose of this paper is to investigate these models in a Hamiltonian language and attempt to obtain a physical understanding of how the anomaly works. Jansen [5] has performed a similar study, and this should be considered as an extension of that work. A preliminary discussion of some of our ideas is contained in Ref. [6].

II. MASSLESS FERMIONS AND THE ANOMALY

Chiral symmetry is intimately tied with Lorentz invariance. A massive particle of spin s has $2s + 1$ distinct spin states which mix under a general Lorentz transformation. The helicity of a massless particle, on the other hand, is frame invariant. Indeed, for free particles one can write down local fields which create or destroy just a single helicity state. For spin $\frac{1}{2}$ fermions coupled minimally to gauge fields, their helicity remains naively conserved.

In one space dimension the roles of left- and right-handed helicities are replaced by left- and right-moving particles. Since an observer cannot go faster than light, he can never overtake a massless particle and a right mover will be so in all Lorentz frames.

The fact that Lorentz invariance is crucial here provides another warning that chiral symmetry on the lattice will be difficult. Indeed, lattice formulations inherently violate the usual space-time symmetries. Chiral issues should only be expected to be useful for states of low energy which are not affected by the underlying lattice structure.

While separate phase rotations of left- and right-handed massless fermions give the formal symmetry of a continuum gauge theory, well known anomalies arise through the divergences of quantum field theory. In particular, there is the famous triangle diagram where a virtual fermion loop couples an axial vector current to two vector currents. This diagram cannot be regulated so that both the vector and axial vector currents are conserved. In two space-time dimensions the analogous problem arises with a simple bubble diagram connecting a vector and an axial vector current.

The fact that the anomaly must exist can be intuitively argued in analogy with band theory in solid state physics. With massive fermions the vacuum has a Fermi level midway in a gap between the filled Dirac sea and a continuum of positive energy particle states. This represents an insulator. As the mass is taken to zero, the gap closes and the vacuum becomes a conductor. External gauge fields applied to this conductor can induce currents. For a specific example, consider a one-space-dimensional world compactified into a ring. A changing magnetic field though this ring will induce currents, changing the relative number of left- and right-moving particles. Without the anomaly, transformers would not work.

In this problem, physics should be periodic in the amount of flux through the ring. This is a one dimensional analogue of the periodicity of four-dimensional non-Abelian gauge theories as one passes through topologically nontrivial configurations [7]. The latter case with the standard model gives rise to nonconservation of the baryon current [3].

With the one dimensional ring, the strength of the field characterizes the phase that a charged particle acquires in running around the ring. As this net phase adiabatically increases, the individual fermionic energy levels shift monotonically. As one adds another unit of flux through the ring, one filled right-moving level from the Dirac sea moves, say, up to positive energy, while one empty left-moving level drops into the sea, leaving a hole.

This induces a net current carried by a right-moving particle and a left-moving antiparticle. This way of visualizing how the anomaly works was nicely discussed some time ago [8]. We will find a similar picture when we investigate the lattice models.

III. THE DOUBLING PROBLEM

The essence of the lattice doubling problem already appears with the simplest fermion Hamiltonian in one space dimension:

$$H = iK \sum_j a^{\dagger}_{j+1} a_j - a^{\dagger}_j a_{j+1} . \tag{1}$$

Here j is an integer labeling the sites of an infinite chain and the a_j are fermion annihilation operators satisfying standard anticommutation relations

$$[a_j, a^{\dagger}_k]_+ \equiv a_j a^{\dagger}_k + a^{\dagger}_k a_j = \delta_{j,k} . \tag{2}$$

The bare vacuum $|0\rangle$ satisfies $a_j|0\rangle = 0$. This vacuum is not the physical one, which contains a filled Dirac sea. We refer to K as the hopping parameter. Energy eigenstates in the single fermion sector

$$|\chi\rangle = \sum_j \chi_j a^{\dagger}_j |0\rangle \tag{3}$$

can be easily found in momentum space:

$$\chi_j = e^{iqj} \chi_0 , \tag{4}$$

where $0 \leq q < 2\pi$. The result is

$$E(q) = 2K \sin(q) . \tag{5}$$

The physical vacuum has the negative energy states filled to form a Dirac sea. This is sketched in Fig. 1.

If we consider a fermionic wave packet produced from small momentum q, then, since the group velocity dE/dq is positive in this region, it will move to the right. On the other hand, a wave packet produced from momenta in the vicinity of $q \sim \pi$ will be left moving. The essence of
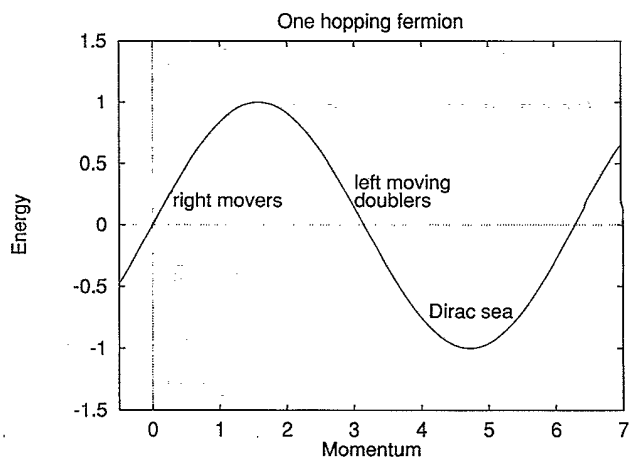

FIG. 1. Spectrum of fermions hopping along a line with the Hamiltonian in Eq. (1).

the Nielsen-Ninomiya theorem [9] is that we must have both types of excitation. The periodicity in q requires the dispersion relation to have an equal number of zeros with positive and negative slopes.

The recent attempts to circumvent this result add to the spectrum an infinite number of additional states at high energy [10]. The idea is to have a mode with $E = 2K \sin(q)$ still exist at small q, but then become absorbed in an infinite band of states before q reaches π. If the band is truly infinite, then the extra state does not have to reappear as the momentum increases to 2π. In the domain wall picture, this infinite tower of states is represented by a flow into the extra dimension.

IV. THE WILSON APPROACH

In this section, we review Wilson's scheme for adding a nonchirally symmetric term to remove the doublers appearing in a naive lattice transcription of the Dirac equation. We do this in some detail because the general behavior of the Wilson-fermion Hamiltonian will be central to our later construction of surface modes. To keep the discussion simple, we work in one dimension with a two component spinor

$$\psi = \begin{bmatrix} a \\ b \end{bmatrix} . \tag{6}$$

The most naive lattice Hamiltonian begins with the simple hopping case of Eq. (1) and adds in the lower components and a mass term to mix the upper and lower components:

$$H = iK \sum_j a_{j+1}^\dagger a_j - a_j^\dagger a_{j+1} - b_{j+1}^\dagger b_j + b_j^\dagger b_{j+1}$$
$$+ M \sum_j a_j^\dagger b_j + b_j^\dagger a_j . \tag{7}$$

Introducing Dirac matrices

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

FIG. 2. Spectrum of naive fermions in one dimension.

and defining $\bar{\psi} = \psi^\dagger \gamma_0$, we write the Hamiltonian more compactly as

$$H = \sum_j iK (\bar{\psi}_{j+1} \gamma_1 \psi_j - \bar{\psi}_j \gamma_1 \psi_{j+1}) + M \sum_j \bar{\psi}_j \psi_j . \tag{9}$$

As before, the single particle states are easily found by Fourier transformation and satisfy

$$E^2 = 4K^2 \sin^2(q) + M^2 . \tag{10}$$

This spectrum is sketched in Fig. 2. Again, we are to fill the negative energy sea.

Naive chiral symmetry is implemented through distinct phase rotations for the upper and lower components of ψ. The mass term mixes these components and opens up a gap in the spectrum. The doublers at $q \sim \pi$, however, are still with us.

To remove the degenerate doublers, we make the mixing of the upper and lower components momentum dependent. A simple way of doing this was proposed by Wilson [11]. In our language, we add one more term to the Hamiltonian:

$$H = iK \sum_j a_{j+1}^\dagger a_j - a_j^\dagger a_{j+1} - b_{j+1}^\dagger b_j + b_j^\dagger b_{j+1} + M \sum_j a_j^\dagger b_j + b_j^\dagger a_j - rK \sum_j a_j^\dagger b_{j+1} + b_j^\dagger a_{j+1} + b_{j+1}^\dagger a_j + a_{j+1}^\dagger b_j$$

$$= \sum_j K (\bar{\psi}_{j+1}(i\gamma_1 - r)\psi_j - \bar{\psi}_j(i\gamma_1 + r)\psi_{j+1}) + \sum_j M \bar{\psi}_j \psi_j . \tag{11}$$

Now the spectrum satisfies

$$E^2 = 4K^2 \sin^2(q) + [M - 2rK \cos(q)]^2 . \tag{12}$$

This is sketched in Fig. 3.

Note how the doublers at $q \sim \pi$ are increased in energy relative to the states at $q \sim 0$. The physical particle mass is now $m = M - 2rK$ while the doubler is at $M + 2rK$.

When r becomes large, the dip in the spectrum of Fig. 3 near $q = \pi$ actually becomes a maximum. This is irrelevant for our discussion, although we note that the
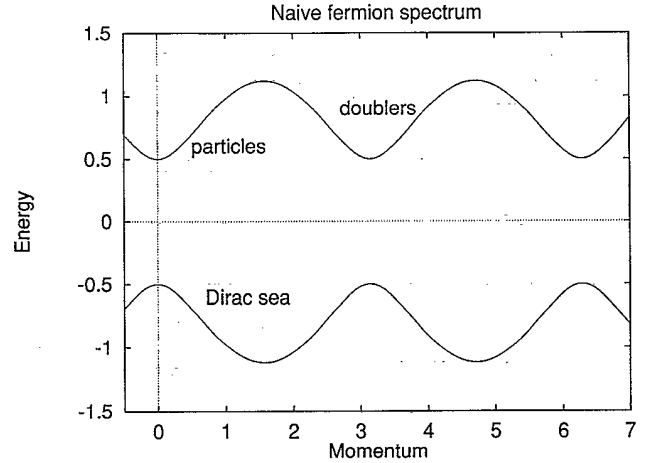
case $r = 1$ is somewhat special. For this value, the matrices $i\gamma_1 \pm r$, which determine how the fermions hop along the lattice, are proportional to projection operators. In a sense, the doubler is removed because only one component can hop. This choice for r has been the most popular in practice, but we remain more general.

The hopping parameter has a critical value at

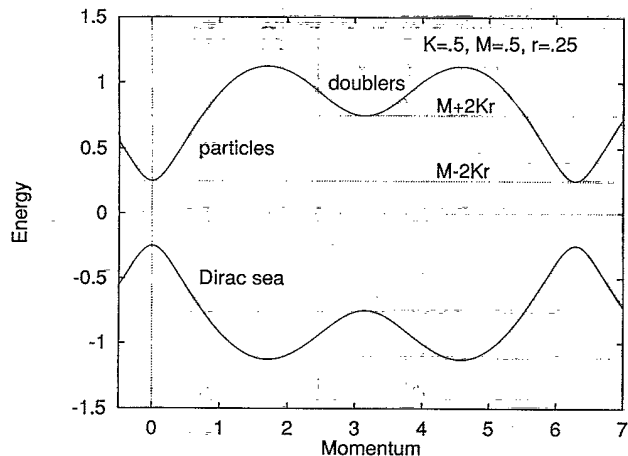$$K_{\text{crit}} = \frac{M}{2r} . \tag{13}$$

FIG. 3. Spectrum of Wilson fermions in one dimension.

At this point the gap in the spectrum closes and one species of fermion becomes massless. This is sketched in Fig. 4. The Wilson term, proportional to r, still mixes the a and b type particles; so there is no exact chiral symmetry. Nevertheless, in the continuum limit this represents a candidate for a chirally symmetric theory. Beforehand, as discussed in Ref. [12], chiral symmetry does not provide a good order parameter.

A difficulty with this approach is that gauge interactions will renormalize the parameters. To obtain massless pions one must finely tune K to K_{crit}, an *a priori* unknown function of the gauge coupling. Despite the awkwardness of such tuning, this is how numerical simulations with Wilson quarks generally proceed. The hopping parameter is adjusted to get the pion mass right, and one hopes for the remaining predictions of current algebra to reappear in the continuum limit.

V. SUPERCRITICAL K AND SURFACE MODES

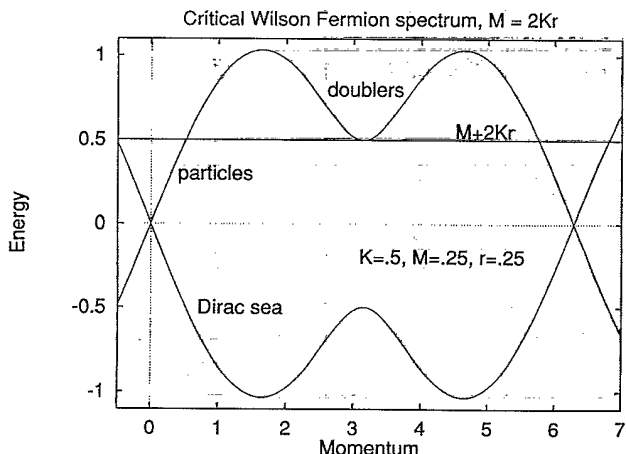The case of K exceeding the critical value $M/2r$ is rarely discussed but quite interesting nevertheless. Aoki

and Gocksch [12] have argued that, as one passes through this point with gauge fields present, there occurs a spontaneous breaking of parity, and, if one has more than one flavor, there is a breaking of flavor symmetry. In their picture one of the pion states becomes massless from the critical behavior at K_{crit}, while the other two remain as Goldstone bosons of the broken flavor symmetry as K increases still further.

Restricting ourselves to the free fermion case for the time being, interesting things happen here for supercritical K as well. As the band closes and reopens with increasing K, the positive energy particle band and the negative energy Dirac sea couple strongly. A similar situation was studied some time ago by Shockley [13], who observed that, if the system is finite with open walls, then two discrete levels leave the bands and emerge bound to the ends of the system. In Fig. 5 we show this phenomenon by plotting the spectrum of states for a box of 20 sites as a function of the hopping parameter.

As the volume of our system goes to infinity, particle-hole symmetry forces these surface levels to go to exactly zero energy. In a finite box, the wave functions have exponential tails away from the walls, mixing the states and in general giving them a small energy.

In the Appendix we prove the general result that there exists such a state bound to any interface separating a region with $K > K_{crit}$ from a region with $K < K_{crit}$. In Ref. [4], Kaplan uses $M = 2Kr + m\epsilon(x)$. We adopt the simpler approach of Shamir [14] and take $K = 0$ on one side, giving modes on an open surface.

The analysis of the Appendix shows the essential nature of the side with supercritical hopping. In our later discussion of the anomaly in terms of a flow into an extra dimension, it will always be a flow into a region of supercritical hopping. This should be contrasted with the continuum discussion of Ref. [15], where the flow is sym-
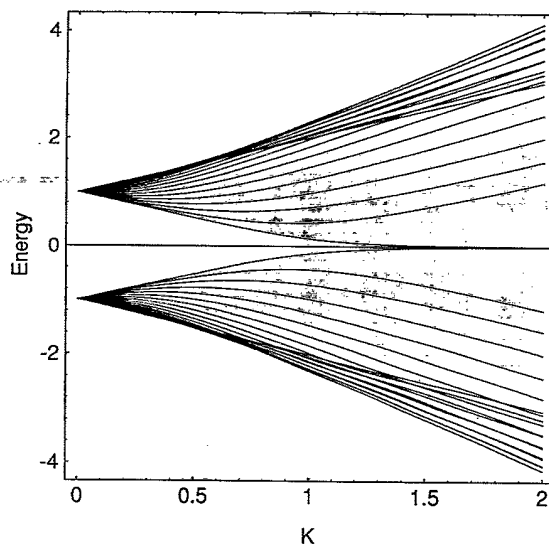

FIG. 4. Spectrum of Wilson fermions at critical hopping. Here we take $K = 0.5$, $M = 0.25$, and $r = 0.25$.

FIG. 5. Energy levels versus hopping parameter K for Wilson fermions on a 20 site lattice. Here we take $M = 1$ and $r = 0.5$. Note the low energy states appearing when $K > K_{crit} = 1$.

metric about the defect. This symmetry appears, however, to be regulator dependent [16]. For example, with a Pauli-Villars regulator, the sign of the fermion mass relative to the regulator mass controls the direction of flow.

Following the usual procedure of filling half the states for the Dirac sea, we see that there is an ambiguity with the last fermion, which could go into either of the degenerate surface modes. If we imagine coupling the fermions to, say, a U(1) gauge field, then this last fermion will be a source of a background electric field which will run to the hole state on the opposite wall. This is the physical origin of the parity breaking proposed in Ref. [12]. In the continuum limit the vacuum should be equivalent to that of the massive Schwinger model with a half unit of background electric flux. The physics of this model in the continuum was discussed in Ref. [17].

VI. EXTRA DIMENSIONS

As the system size goes to infinity, the surface modes quite naturally go to zero energy. This behavior forms the basis for a theory of chiral fermions. The approach of Kaplan [4] is to reinterpret the coordinate labeled by j in the above discussion as an extra dimension beyond the usual space and time. Our physical world exists on a four-dimensional interface, with the light quarks and leptons being the above surface modes.

To be concrete, let us consider adding D space dimensions to the above Hamiltonian, where for the following D will be either 1 or 3. For simplicity we will take L^D space sites and use antiperiodic boundary conditions for each of these dimensions. The extra dimension, which we refer to as the fifth, has L_5 sites and open boundaries. We take the same hopping and Wilson parameters in each of the dimensions, including the fifth, although this is not essential.

Our Dirac matrices γ_μ satisfy the usual

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \ . \tag{14}$$

We define $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ for $D=3$ and $\gamma_5 = \gamma_0\gamma_1$ for $D=1$. We take γ_0 and γ_5 to be Hermitian, while the spatial γ matrices are anti-Hermitian. The Hamiltonian we are led to is then

$$H = \sum_{\mathbf{n},j} \left[K\bar{\psi}_{\mathbf{n},j+1}(\gamma_5 - r)\psi_{\mathbf{n},j} - K\bar{\psi}_{\mathbf{n},j}(\gamma_5 + r)\psi_{\mathbf{n},j+1} \right.$$
$$+ \sum_{a=1}^{D} [K\bar{\psi}_{\mathbf{n}+\mathbf{e}_a,j}(i\gamma_a - r)\psi_{\mathbf{n},j}$$
$$-K\bar{\psi}_{\mathbf{n},j}(i\gamma_a + r)\psi_{\mathbf{n}+\mathbf{e}_a,j}]$$
$$\left. + M\bar{\psi}_{\mathbf{n},j}\psi_{\mathbf{n},j} \right] \ . \tag{15}$$

Here $\mathbf{n}$ denotes the spatial sites, and $\mathbf{e}_a$ is the unit vector in the positive ath direction.

The use of antiperiodic boundary conditions allows us to go to momentum space for the spatial coordinates. Denoting the components of the momentum by q_a, we write

$$\psi_{\mathbf{q},j} = \frac{1}{L^{D/2}} \sum_{\mathbf{n}} e^{-i\mathbf{q}\cdot\mathbf{n}} \psi_{\mathbf{n},j} \ . \tag{16}$$

Each component of the momentum takes discrete values from the set $(2k+1)\pi/L$, where k runs from, say, 0 to $L-1$. This makes our Hamiltonian block diagonal, with each value for $\mathbf{q}$ representing a separate block. In this way the Hamiltonian reduces to

$$H = \sum_{\mathbf{q},j} \left[K\bar{\psi}_{\mathbf{q},j+1}(\gamma_5 - r)\psi_{\mathbf{q},j} - K\bar{\psi}_{\mathbf{q},j}(\gamma_5 + r)\psi_{\mathbf{q},j+1} \right.$$
$$+ \sum_a 2K \sin(q_a)\bar{\psi}_{\mathbf{q},j}\gamma_a\psi_{\mathbf{q},j}$$
$$\left. + \left[M - 2Kr \sum_a \cos(q_a) \right] \bar{\psi}_{\mathbf{q},j}\psi_{\mathbf{q},j} \right] \ . \tag{17}$$

As follows from the discussion in the Appendix, modes bound to the surface in the fifth direction exist whenever K exceeds the critical value

$$K_{\text{crit}} = \frac{M}{2r} - K \sum_a \cos(q_a) \ . \tag{18}$$

Note how this critical value now depends on the spatial momentum. Appropriately choosing M, we can have the surface states exist for small q, but not when any component of $q_a \sim \pi$, thus avoiding doublers [18]. Specifically, when the hopping is direction independent, we want (assuming K, r, and M are all positive)

$$(D-1)K < M/2r < (D+1)/K \ . \tag{19}$$

For convenience we denote $\tilde{M} = M - 2KrD$ and, if not specified otherwise, we use in the illustrative figures that follow the typical set of parameters $\tilde{M} = 0.1$, $K = 0.6$, and $r = \frac{11}{12}$. In Fig. 6 we sketch the spectrum of the Hamil-
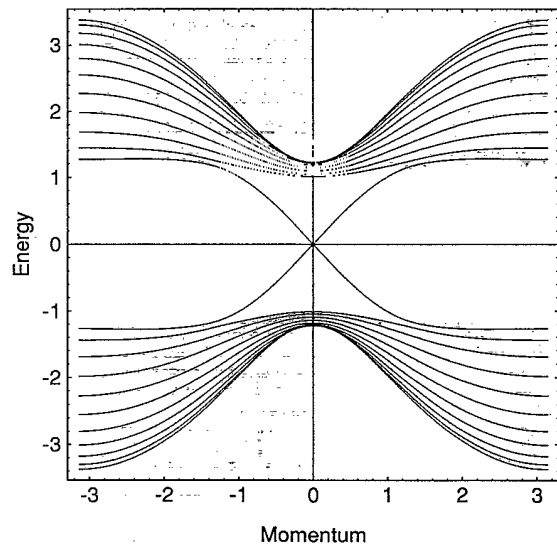

FIG. 6. Energy spectrum as a function of the physical momentum on a lattice with $L_5 = 11$ and $L = 500$. Note the crossing surface modes at low energy. Because of the large spatial size of the lattice, the momentum appears to be almost a continuous variable.

tonian (17) in one space dimension, showing that we indeed avoid doublers in this case.

The fact that the low energy states are bound to the surface of the system is easily seen by studying the expectation value for the fifth coordinate in the single particle states. In Fig. 7 we show the energies of the various levels as a function of this expectation on an $L = 500$ by $L_5 = 11$ lattice.

The above discussion shows that on a single surface we have an elegant lattice theory for a low energy chiral fermion. We would now like to add gauge fields. Here we adopt the attitude that we do not want a lot of new degrees of freedom, and follow Ref. [10] in regarding the extra dimension as a flavor space. In particular, we do not put gauge fields in the fifth dimension, and the physical gauge fields are independent of this dimension.

While this approach has the advantage of preserving an exact gauge invariance and not introducing unwanted fields, it has the disadvantage that both walls are coupled equally to the gauge field. Thus, even when the size of the fifth dimension approaches infinity, the opposite chirality fermions do not decouple. The main thing that has been accomplished so far is to find a theory of fermions coupled in a vectorlike manner, without any doublers, and with a natural way to take the fermion masses to zero.

Later we will briefly discuss decoupling the states on one wall in a gauge noninvariant way. This may give rise to a theory of chirally coupled fermions, but the role of anomaly cancellation remains unclear.

VII. THE ANOMALY AND ROTATING EIGENVALUES

If one's goal is to formulate the massless fermion theory in $2n$ dimensions (chiral or vectorlike) using the surface low energy states of the massive vectorlike theory in $2n + 1$ dimensions, then the question of anomalies is of
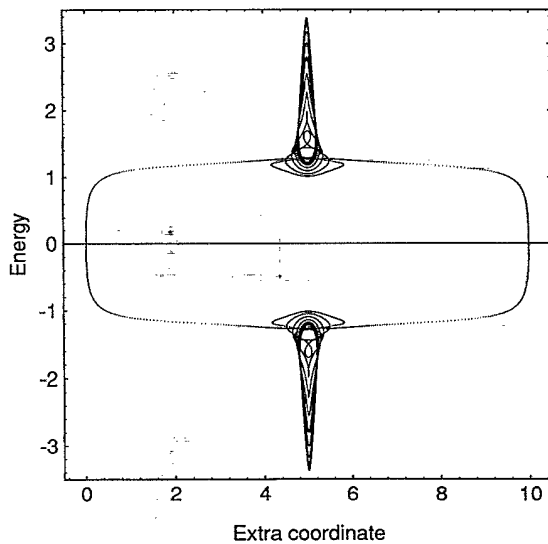

FIG. 7. Energy spectrum as a function of the average position in the extra dimension on an $L_5 = 11$ by $L = 500$ lattice. Note how the low energy modes lie near the ends of the lattice.

primary concern. The reason is quite obvious: If we look at one interface with one chiral fermion existing on it, then we should see the anomaly in gauge current. On the other hand, the full theory we started from is anomaly-free. Therefore, one should not only make sure that the correct anomaly is reproduced, but also understand the mechanism of how this happens.

The basic scenario that clarifies the situation was discussed in a somewhat different context by Callan and Harvey [15]. They consider a vector theory, whose mass term has a domain wall shape in an extra dimension, and show that it has a chiral zero mode existing on the wall. The anomalous gauge current generated by this zero mode has to be canceled in the underlying $(2n + 1)$-dimensional theory since that world is anomaly-free. Indeed, the massive modes contribute to the low energy effective action a piece representing the flow of charge into (or our of) the wall from the extra dimension. When calculated far from the wall, it cancels the anomalous contribution. In the U(1) case in $2+1$ dimensions this was recently explicitly checked on the lattice with both Kaplan's and Shamir's formulations [19]. Indeed the cancellation is valid even close to the wall [20]. Therefore what on the interface looks like an anomaly is the flow of charge into the extra dimension and the role of the heavy modes is to carry that charge.

The above picture was adopted by Kaplan in his lattice proposal [4] and in Ref. [21] Shamir carried out a detailed study of the anomaly in this scheme. Here we explicitly visualize how this works with our Hamiltonian. We already mentioned that at any finite L the model is strictly vectorlike with our gauge prescription. Nevertheless, since opposite chirality partners live on opposite walls, the charge still has to be transported through the extra dimension and we should see exactly what Callan and Harvey suggested. We will examine in detail how the heavy modes behave in external fields, causing the anomaly. As is often the case, the Hamiltonian formalism can offer more physical intuition than the (often practically more powerful) Lagrangian approach. Since we want to see the heavy modes "in action," we work with the full fermion theory and study how the one particle states respond to a particular external field.

For simplicity, we concentrate on the one-dimensional case with gauge group U(1). If we have an external U(1) gauge field, it is manifested in terms of phase factors whenever a fermion hops from one site to the next. Calling this factor $U(k) = e^{i\alpha_k}$ for the hopping from space site k to $k + 1$, the full Hamiltonian becomes

$$H = \sum_{k,j} [K \bar{\psi}_{k,j+1}(\gamma_5 - r)\psi_{k,j} - K \bar{\psi}_{k,j}(\gamma_5 + r)\psi_{k,j+1}$$

$$+ K \bar{\psi}_{k+1,j} U(k)(i\gamma_1 - r)\psi_{k,j}$$

$$- K \bar{\psi}_{k,j} U^\dagger(k)(i\gamma_1 + r)\psi_{k+1,j} + M \bar{\psi}_{k,j}\psi_{k,j}] .$$

$$(20)$$

A gauge transformation by a phase g at site k takes $U(k)$ to $U(k)g^\dagger$, $U(k-1)$ to $gU(k-1)$, and ψ_k to $g\psi_k$. The invariance of the Hamiltonian under this symmetry

tells us that the spectrum only depends on the product of all the $U(k)$'s. This is the net phase acquired by a fermion in traveling all the way around our finite periodic system.

A particularly convenient gauge choice is to evenly distribute the phases so that $U(k)=e^{i\alpha}$ is independent of k. This keeps momentum space simple, with the Hamiltonian becoming

$$H=\sum_{q,j}\{K\bar{\psi}_{q,j+1}(\gamma_5-r)\psi q,j-K\bar{\psi}_{q,j}(\gamma_5+r)\psi_{q,j+1}$$
$$+2K\sin(q-\alpha)\bar{\psi}_{q,j}\gamma_1\psi_{q,j}$$
$$+[M-2Kr\cos(q-\alpha)]\bar{\psi}_{q,j}\psi_{q,j}\} \ . \qquad (21)$$

The energy eigenstates are functions of the momentum shifted by α. As α increases by $2\pi/L$, sequential momenta rotate into each other. The total net phase in this case is 2π and, as expected, physics goes back to itself. In what follows we will often refer to α in its natural units, calling $2\pi/L$ one unit of flux.

In the adiabatic limit, the time evolution is a continuous change of one particle states with changing α. It is therefore sufficient to consider the spectrum flow with respect to α itself. In Fig. 8 we show the positions of one particle states (mean values of x_5) for the values $\alpha=0,\frac{1}{4},\frac{1}{2},\frac{3}{4}$. One can see quite transparently what is happening; the low energy states at the lattice ends change energy without substantially changing their position in the extra dimension. The same is true for the very high energy states, residing deep in the lattice interior. However, the surface states with energies close to the cutoff are very sensitive to the applied field. When the energy of such a level gets close to the cutoff, it moves swiftly towards the middle. At the same time, another level from the middle lowers its energy and runs towards the opposite wall. This is also true for corresponding lev-
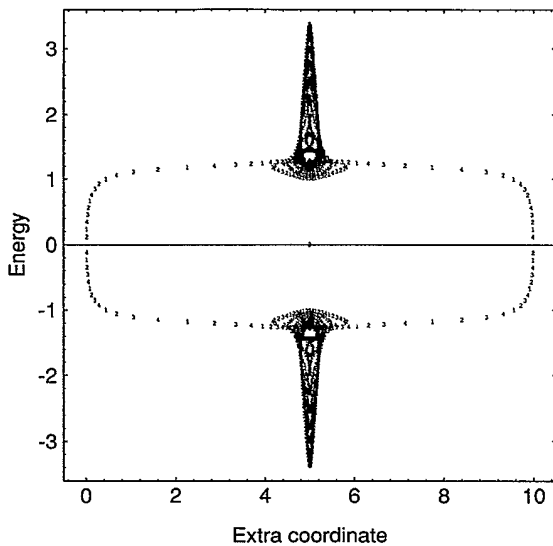
els with negative energy at the cutoff; they just move in the opposite direction. Therefore, we see how the heavy modes right at the cutoff are responsible for carrying the charge on and off the end surfaces. Consider applying the gauge field to the physical vacuum with all negative energy states filled. Then these "flying states" are responsible for what appears to be the gauge anomaly on the surfaces.

To make our discussion more formal and quantitative, we now compute the anomaly in the axial charge of our vectorlike model. To do that, we first give a simple definition of the axial charge. On the lattice there is considerable freedom here; any definition assigning opposite charges to the two zero modes living on the opposite walls should yield a correct continuum limit. In our model, the most natural measure for "being chiral" is in fact the measure of "being on the wall." Therefore we define the operator of the axial charge to be the fermion number weighted by the location in the extra dimension:

$$Q_5=\frac{1}{L_5-1}\sum_{q,j}(L_5-1-2j)\psi_{q,j}^\dagger\psi_{q,j} \ , \qquad (22)$$

with $j=0,1,2,\ldots,L_5-1$. This assigns to a one particle state $+1$ if it is exactly bound to the left wall with $j=0$ and -1 if it is bound to the right wall. For states smeared uniformly over j, we obtain zero. Regarding the extra dimension as an internal space, we see that the axial charge is nothing but a particular combination of flavor charges.

We define the vacuum of the theory in the usual way by filling all the negative energy eigenstates. We then measure the energy of excited states relative to this state. At zero field, the vacuum has zero axial charge because of the mirror symmetry of the Hamiltonian with respect to the middle of the extra space $(j\to L_5-1-j)$. To see the anomaly, we evolve the vacuum in an adiabatic field, increasing the value of α from 0 to 1 unit of flux, and look for the change in the total axial charge.

What we should see if the theory is to have the correct anomaly structure can be deduced from the continuum expression for chiral anomaly in two dimensions: namely,

$$\Delta Q_5(t)=\frac{e}{2\pi}\int_0^t dt'\int dx\ \epsilon_{\mu\nu}F^{\mu\nu}(x,t')$$
$$=-\frac{e}{2\pi}L\int_0^t 2\dot{A}_1(x,t')=-2e\frac{L}{2\pi}A_1(t) \ . \qquad (23)$$

In our units and with identification $\alpha=eA$ we have

$$\Delta Q_5(t)=-2\alpha(t) \qquad (24)$$

and what we should see is a straight line with the slope of -2.

In Fig. 9 we show that this is indeed what happens in our model. As the field is turned on, the levels in the Dirac sea start to move anticlockwise $(\alpha>0)$, decreasing the total axial charge to negative values in an almost strictly linear manner. In fact, a least squares fit yields the slope -1.995 in this particular case with negligible admixture of higher powers.

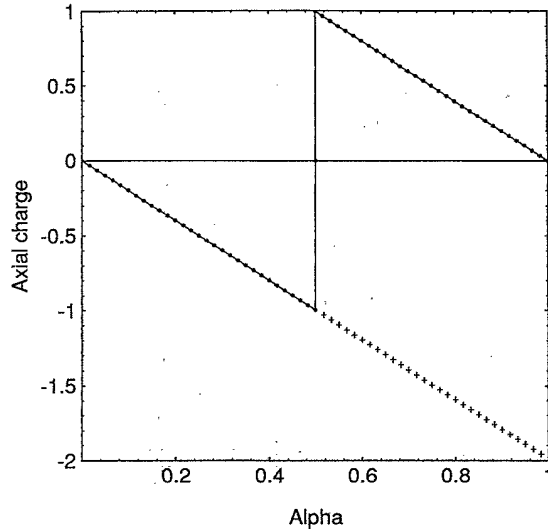When the value of the field is close to one-half unit of

FIG. 8. Energy spectrum as a function of $\langle x_5\rangle$ for $\alpha=0,\frac{1}{4},\frac{1}{2},\frac{3}{4}$ on an $L_5=11$ by $L=16$ lattice. Successive positions of energy levels are marked by 1, 2, 3, and 4, respectively. The levels rotate in the anticlockwise sense.

FIG. 9. Total axial charge of the vacuum as a function of α. The connected line represents the evolution for the time scale of turning on the gauge field τ, being much longer than the time scale δ set by the particle tunneling through the extra dimension. The continuation by crosses applies for $0 \ll \tau \ll 1/\delta$.

flux, one of the filled levels on the right surface is just about to become the positive energy particle and one empty level on the left is just about to drop into the sea. However, as long as our extra dimension is finite, the surface states are not exactly massless. There is always a tiny mass δ present, caused by mixing of the states on the opposite walls. When the fields are truly adiabatic, with the typical time τ for turning on the fields longer than any other time scale in the problem, e.g., $\tau \gg 1/\delta$, instead of creation of a particle-hole pair, the two levels will have enough time to exchange between the walls. As a consequence, we observe a jump at $\alpha = \frac{1}{2}$, which allows vacuum to evolve into its original state as α approaches one unit of flux. In Fig. 8 this is represented by the two almost degenerate levels with energies $\pm \delta$, located in the middle of the pattern. Although the total change of the axial charge is zero in this case, the anomaly can still be extracted from the slope for fields smaller than one-half unit. Note also that, for all practical purposes, the time scale $1/\delta$ is so huge (in our case we have $\delta \approx 10^{-9}$ with just 11 sites in the extra dimension) that only an extremely slow turning on of the field can be rightfully considered adiabatic. With a faster turn on, a particle-hole pair is created on opposite walls, and at one unit of flux the net change in Q_5 approaches -2, as it should. In Fig. 9 this behavior is represented by pluses continuing on the straight line.

VIII. CHIRAL MODELS WITH BROKEN GAUGE INVARIANCE

In an attempt to construct a truly chiral model we would like to decouple the zero modes located on the opposite walls. The common gauge field shared by the two walls couples the chiral partners directly to each other.

To avoid this, we would like the gauge fields to be experienced by one of the zero modes but not by the other; we would like to switch off the gauge field as one traverses the extra dimension.

The problem is that when the gauge field on a given link at the jth slice in the extra dimension is different from the value at the slice $(j+1)$, local gauge invariance is broken at that particular place. Any "switching off" procedure is therefore unavoidably accompanied by an explicit gauge symmetry breaking in the full theory on a finite lattice.

There are many ways to turn off the field. One natural choice is singled out by the requirement that the gauge symmetry is broken in the smallest region possible, namely, only between two slices in the extra dimension. In this way we are led to study the model, consisting of $N = N_1 + N_2$ slices in extra dimension, with N_1 slices coupled to the same gauge field as before and with zero field on the remaining N_2 slices. In what follows we consider even numbers of sites in the extra dimension and the symmetric case $N_1 = N_2 = N/2$.

This scheme is closely related to that used in [22]. In that paper, an additional scalar field was introduced to restore a local gauge invariance. This was analyzed in the small field regime, where it was found that new low mass modes were bound to the location of the gauge field shutoff in the extra dimension. Here we consider effectively freezing this scalar field to a unit value.

Another possibility would be to gradually turn off the gauge field as we pass through the extra dimension. This corresponds to letting the gauge field leak away as one goes deeper into the slab. Our results with such a scheme were qualitatively similar to what we show below for a sharp cutoff. On the other hand, it is possible that ambiguities might arise with a gradual cutoff of non-Abelian fields.

This picture rather inelegantly spreads a breaking of exact gauge invariance over the extra dimension. It is unclear how much damage is done by this sacrifice. Indeed, we would be happy if the net result were a well regulated theory of a single chiral fermion. The theory seems to behave as desired when classical external fields are considered, as we will see implicitly below. Nevertheless, there are plausibility arguments that, when the gauge fields are dynamical, gauge invariance may be dynamically restored with the theory becoming vector-like again (see [14] and references therein).

Here we do not fully resolve this question. Numerical simulations may be necessary to see in detail what is going on. Instead, we point out one amusing possibility that this gauge variant model seems to offer, namely, that it gives rise to a natural definition of a winding number for classical gauge configurations on a lattice. This involves counting the quantum one-fermion states of the gauge variant system in a given external gauge field. In that respect our approach resembles the discussion in Ref. [10].

The basic idea is as follows. Assume that the field resides on the left half of the slab, while on the right half the field is permanently turned off. For a given external field, we can diagonalize the fermion Hamiltonian to ob-

tain a set of one particle levels. If we start from zero field and increase the field gradually, one expects the levels located on the left to rotate in a similar way to that seen in the previous section for the gauge invariant model. Eventually, one positive level will cross zero and become a negative energy level. On the other hand, the levels on the right will not experience these changes and are not expected to rotate. The crossing level on the left is not compensated by a crossing partner (in the opposite direction) on the right and as a net result we have one more level in the Dirac sea. Increasing the field further, more levels will go down, while switching the field to negative values should do the opposite: expel the states out of the sea. In this way, we divide the gauge configurations into classes, each class labeled by the "excess of levels in the Dirac sea."

In Figs. 10 and 11 we demonstrate that the gauge variant model indeed behaves as described above. Figure 10 shows the shift of levels as the field is turned on from zero value to $-\frac{1}{2}$ unit. The low energy levels located on the left move up and one of them is just about to escape the Dirac sea. The levels on the opposite wall, on the other hand, stay unchanged and this behavior is to be compared with the one in the gauge invariant model (see Fig. 8). Figure 11 shows clearly how the vacuum is deformed when -1 unit of flux is turned on, expelling one right-moving level.

Formally, we can define the winding number by

$$n = \tfrac{1}{2}(N_- - N_+) \, , \tag{25}$$

where N_+, N_- is the number of positive and negative levels respectively. In Fig. 12 we show that with this definition the expected staircase assignment of the winding number really takes place.
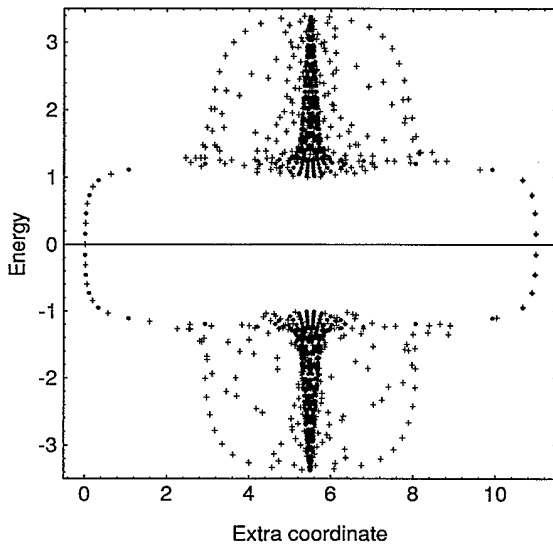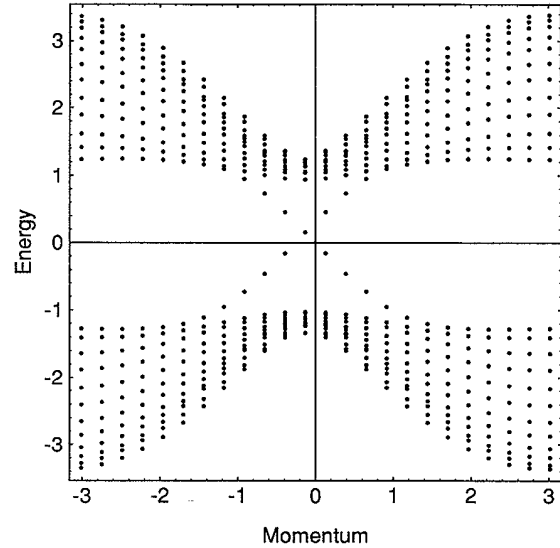

FIG. 11. Energy spectrum as a function of momentum for the field equal to -1 unit of flux, sharply switched off at the middle of the extra space. We show the results on an $L5=12$ by $L=24$ lattice, indicating that one level (right mover) escaped the Dirac sea.

Note that, while their behavior is rather complex, none of the levels deep in the middle of the extra dimension drops to low energy with these small values for the field. With our strong breaking of gauge invariance, we avoid the low energy states seen in Ref. [22] bound to the place where the gauge field is cut off.

Except for exceptional cases where an eigenvalue exactly vanishes, this definition is always a well defined integer. Note that this is true regardless of whether the gauge field represents a gauge copy of the vacuum or not. An instanton in this language would occur whenever a
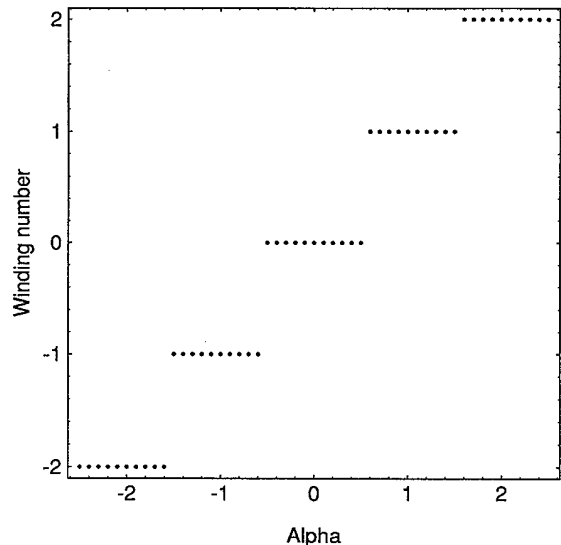

FIG. 10. Energy spectrum as a function of x_5 for zero field (black points) and for $\alpha = -\frac{1}{2}$ (pluses), sharply switched off at the middle of the extra space. In this case we consider an $L_5=12$ by $L=24$ lattice. The low energy states on the wall with nonzero field move up while those on the opposite wall stay unchanged.

FIG. 12. Winding number as defined in the text for several gauge field configurations (values of α). The staircase assignment takes place as expected.

gauge field configuration interpolates in time between two spatial vacuum configurations of different winding number [7].

The correspondence with the continuum case is immediate. Consider continuum massless electrodynamics on a circle with circumference L and pick the gauge $A_0(x,t)=0$, $A_1(x,t)=A_1(t)$. The periodic gauge transformations with winding number n are specified by $\Lambda = e^{inx2\pi/L}$ and the pure gauge configurations generated by these functions are $A_1 = n2\pi/L = -i\Lambda^{-1}(d/dx)\Lambda$. This is what we see for integer α in Fig. 12. It might be interesting to compare this definition of winding number with other prescriptions, i.e., Ref. [23].

IX. WEAK INTERACTIONS, MIRROR FERMION MODEL

With an exact gauge invariance and a finite size for the extra dimension, the surface models are inherently vectorlike. The fermions always appear with both chiralities, albeit separated in the extra dimension. However, experimentally we know that only left-handed neutrinos couple to the weak bosons. In this section we discuss one way to break the symmetries between these states, resulting in a theory with only one light gauged chiral state. Here we keep the underlying gauge symmetry exact, but do require that the chiral gauge symmetry be spontaneously broken, just as observed in the standard model. The picture also contains heavy mirror fermions. If anomalies are not canceled amongst the light species, these heavy states must survive in the continuum limit. It remains an open question when anomalies are properly canceled whether it might be possible to drive the heavy mirror states to arbitrarily large mass.

We start by considering two separate species ψ_1 and ψ_2 in the surface mode picture. However, we treat these in an unsymmetric way. For ψ_1 use our previous Hamiltonian. For ψ_2 we change the sign of all terms proportional to γ_5. On a given wall, the surface modes associated with ψ_1 and ψ_2 will then have opposite chirality.

Now we introduce the gauge fields. Since we want to eventually couple only one-handed neutrinos to the vector bosons, consider gauging ψ_1 but not ψ_2. Indeed, at this stage ψ_2 represents a totally decoupled right-handed fermion on one wall. We still have a mirror situation on the opposite wall, consisting of a right-handed gauged state and a left-handed decoupled fermion.

The next ingredient is to spontaneously break the gauge symmetry, as in the standard model, by introducing a Higgs field ϕ with a nonvanishing expectation value. We can use this field to generate masses as in the standard model by coupling ψ_1 and ψ_2 with a term of the form $\bar{\psi}_1\psi_2\phi$.

The new feature is to allow the coupling to the Higgs field to depend on the extra coordinate. In particular, let it be small or vanishing on one wall and large on the other. The surface modes are then light on one wall and heavy on the other.

In Fig. 13 we consider one space dimension and sketch the fermion spectrum of this model with vanishing gauge fields and a constant Higgs field. With an increasing
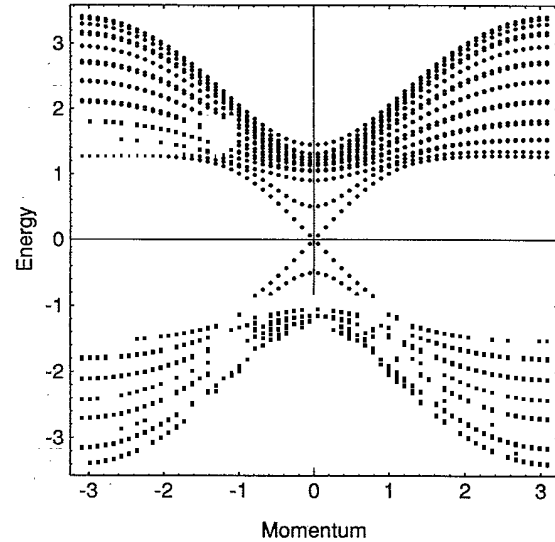

FIG. 13. Energy levels versus momentum for the two species model discussed in the text. In this case, we take an $L_5 = 10$ by $L = 60$ lattice and switch off the coupling between species sharply at the middle of the extra space. Note how one species is massless and the other massive.

gauge field as considered in the previous section, only one component of the light fermions will shift. As the flux approaches one-half unit, however, something new happens. Now the Higgs field will also be affected, and it can become favorable for it to acquire a winding to reduce the system energy. As the flux approaches a full unit, then the physics of the model returns to a gauge transform of the original vacuum.

This model is quite closely related to the picture of the previous section. There we could have restored the gauge invariance with a Higgs field, as was done in Ref. [22]. Then the model is essentially equivalent to that considered here. The difference in the last section was that we froze the Higgs field and did not allow the above unwinding to occur. This broke the gauge symmetry, and the physics with a full unit of flux is not equivalent to the gauge invariant situation discussed here. On the other hand, the explicit breaking was essential to the definition of winding number.

As in other mirror fermion models [24], triviality arguments suggest that there might exist bounds on the mass of the heavy particles. This is certainly expected to be the case where the light fermions alone give an anomalous gauge theory, in which case we expect the mirror particles cannot become much heavier than the vector mesons, i.e., the W.

It is conceivable that the restrictions on the mirror fermion masses are weaker when anomalies cancel amongst the light states. In this case there is no perturbative need for the heavy states, and perhaps they can be driven to infinite mass in the continuum limit. This is a rather speculative desire, but, if so, we would have a candidate for a lattice discretization of the standard model.

Unfortunately, the model as it stands does not lead to

baryon number violation [25]. The anomaly will involve a tunneling of baryons from one wall to the opposite, where they become mirror baryons. Even if these extra particles are heavy, the decay can only occur through mixing with the ordinary particle states. In this sense, the mirror particles still show their presence in low energy physics. This further hints that the mirror fermions might not be removable in the continuum limit.

Another speculative proposal is to use the right-handed mirror states in some way as observed particles. Indeed, the world has left-handed leptons and right-handed antibaryons. Any simple extension of this idea to a realistic model must unify these particles [26]. On the other hand, the fact that the anomalies are canceled between different representations of the SU(3) of strong interactions may preclude such options.

X. CONCLUSIONS

We have studied the use of Shockley surface states as the basis of a theory of chiral fermions. For strong interaction physics this yields an elegant formulation where the massless limit for the quarks is quite natural.

We have seen explicitly how the anomaly works in terms of a flow in the extra dimension used to formulate the model. For anomaly-free currents, the net flow in this dimension cancels, and we expect the predictions of current algebra to arise naturally. On the other hand, the symmetries for singlet axial vector currents are strongly broken by this flow. This presumably precludes the need for a corresponding Goldstone boson and solves the U(1) problem.

Several questions remain before we have a theory of the weak interactions on the lattice, where the gauge fields are to be coupled to chiral currents. We seem to be led to a theory with mirror fermions on the opposing walls of the system. In a spontaneously broken theory these extra states can be given different masses. Whether they can be driven to infinite mass in the continuum limit presumably depends on whether all necessary chiral anomalies have been canceled.

APPENDIX: CONDITIONS FOR ZERO MODES

In this Appendix we prove the general result that a fermion zero mode will be bound to any defect separating a region with supercritical hopping from a region with subcritical hopping. We restrict ourselves to a single spatial momentum q from the full Hamiltonian of Eq. (17), and look for single particle states of the form

$$|\chi\rangle = \sum_j \psi_j^\dagger \chi_j |0\rangle \ . \tag{A1}$$

Here χ represents a spinor of ordinary numbers (i.e., it does not anticommute with anything), and the vacuum $|0\rangle$ is the state annihilated by the operator ψ.

We are interested in eigenstates satisfying the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle \ . \tag{A2}$$

We later consider position dependent values for the parameters K, M, and r, but for now consider working in a region where they are constant. Since our Hamiltonian is then translation invariant, we first study solutions of the form

$$\chi_j \sim \lambda^j \chi_0 \tag{A3}$$

with λ being a general complex number. For plane waves we would restrict ourselves to $\lambda = e^{iq}$; however, as we later plan to match exponential solutions across defects, we remain more general.

Putting this all together, we find that the eigenvalues and the wave function satisfy

$$\left[\not{p} + K \left[\lambda - \frac{1}{\lambda} \right] \gamma_5 + rK \left[\lambda + \frac{1}{\lambda} \right] \right.$$
$$\left. - \left[M - 2Kr \sum_a \cos(q_a) \right] \right] \chi_0 = 0 \ , \tag{A4}$$

where we have defined $p_0 = E$, $p_a = 2K \sin(q_a)$, and the usual $\not{p} = p_0 \gamma_0 - \sum_a p_a \gamma_a$.

Multiplying Eq. (A4) by

$$\not{p} + K \left[\lambda - \frac{1}{\lambda} \right] \gamma_5 - rK \left[\lambda + \frac{1}{\lambda} \right] + \left[M - 2Kr \sum_a \cos(q_a) \right]$$

tells us that any nontrivial solution must satisfy

$$p^2 + K^2 \left[z^2 - 4 - \left[M/K - rz - 2r \sum_a \cos(q_a) \right]^2 \right] = 0 \ , \tag{A5}$$

where we define $z = \lambda + 1/\lambda$. Note that, for any solution λ, $1/\lambda$ also is. Thus for every exponentially increasing solution, there is another which exponentially decreases.

We are ultimately interested in chiral solutions representing massless particles. This suggests we look for states satisfying

$$\not{p}\chi = 0 \ . \tag{A6}$$

For vanishing spatial momentum this is equivalent to $E = 0$. This restriction simplifies the equations dramatically. Indeed, Eq. (A4) then implies that χ_0 is an eigenvector of γ_5. Since this matrix only has eigenvalues ± 1, we conclude that

$$\pm K(\lambda - 1/\lambda) - M + Kr \left[\lambda + 1/\lambda + 2 \sum_a \cos(q_a) \right] = 0 \ , \tag{A7}$$

which is a simple quadratic equation for λ. For a particular set of parameters we show the solutions for λ as a function of the hopping parameter K in Fig. 14. Some qualitative details of this picture apply only to small r, but a similar discussion applies in other cases.
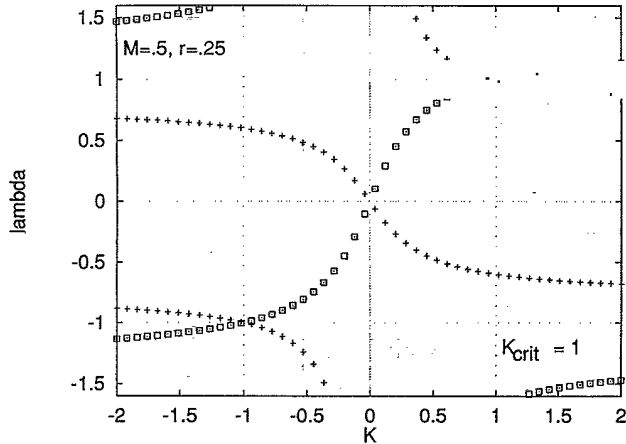
FIG. 14. Translation eigenvalues at zero energy versus hopping parameter. Note the crossing at the critical hopping $K = 1$. Here we have taken $r = 0.25$ and $M - 2Kr \sum_a \cos(q_a) = 0.5$. The pluses correspond to $\gamma_5 \chi = \chi$ and the boxes to $\gamma_5 \chi = -\chi$.

Note the crossing with $\lambda = 1$ in Fig. 14 at the critical hopping. This is a crucial point to the following. When K exceeds the critical value, both solutions which satisfy $\gamma_5 \chi = \chi$ have $|\lambda| < 1$. Conversely, the two increasing wave functions have the opposite chirality. This means that, when $K > K_{\text{crit}}$, if we can ensure that a wave function satisfies the chiral relation $\gamma_5 \chi = \chi$, it must exponentially decrease as j increases. We now discuss how to guarantee that this chiral condition is preserved.

Consider a defect region where the parameters vary with position. Indexing M by its respective site and K and r by the site on their left, the Schrödinger equation reads

$$K_j (\gamma_5 + r_j) \chi_{j+1} - K_{j-1} (\gamma_5 - r_{j-1}) \chi_{j-1}$$

$$+ \left[\not{p} - M_j + 2 K_j r_j \sum_a \cos(q_a) \right] \chi_j = 0 . \quad (A8)$$

Given χ on two adjacent sites, this determines its value on the next site. Thus we can iteratively propagate the wave function from a pair of neighboring sites to any other location. Suppose we start somewhere with a wave function which satisfies $\not{p}_j \chi_j = 0$ and $\gamma_5 \chi_j = \pm \chi_j$. The important point is that Eq. (A8) implies that both these properties are propagated as we move through the lattice.

To be explicit, consider the case where at large negative j the hopping is subcritical, for large positive j it is supercritical, and there is some arbitrary transition region between. If for large negative j we start off with the exponentially increasing solution with $\not{p} \chi_j = 0$ and $\gamma_5 \chi_j = \chi_j$, then this solution can only couple to the eigenvalues with the same chiral property at large positive j. As discussed above, these are both exponentially decreasing. Such a solution will be automatically normalizable.

The above argument shows that at least one state must exist bound to a region separating subcritical from supercritical hopping. There could in principle exist more, for example, if the intervening region contains several domain walls.

[1] S. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebra and Anomalies* (World Scientific, Singapore, 1985).

[2] D. Petcher, in *Lattice '92*, Proceedings of the International Symposium, Amsterdam, The Netherlands, edited by P. van Baal and J. Smit [Nucl. Phys. B (Proc. Suppl.) **30**, 50 (1993)].

[3] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).

[4] D. Kaplan, Phys. Lett. B **288**, 342 (1992); M. Golterman, K. Jansen, and D. Kaplan, *ibid.* **301**, 219 (1993).

[5] K. Jansen, Phys. Lett. B **288**, 348 (1992).

[6] M. Creutz and I. Horvath, in *Lattice '93*, Proceedings of the International Symposium, Dallas, Texas, edited by T. Draper [Nucl. Phys. B (Proc. Suppl.) **34** (1994)].

[7] M. Creutz, I. Muzinich, and T. Tudron, Phys. Rev. D **17**, 531 (1979).

[8] J. Ambjorn, J. Greensite, and C. Peterson, Nucl. Phys. **B221**, 381 (1983); B. Holstein, Am. J. Phys. **61**, 142 (1993).

[9] H. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981); **B193**, 173 (1981).

[10] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993); Nucl. Phys. **B412**, 574 (1993); Phys. Rev. Lett. **71**, 3251 (1993).

[11] K. Wilson, in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).

[12] S. Aoki, Nucl. Phys. **B314**, 79 (1989); S. Aoki and A. Gocksch, Phys. Rev. D **45**, 3845 (1992).

[13] W. Shockley, Phys. Rev. **56**, 317 (1939); see also F. Seitz, *The Modern Theory of Solids* (McGraw-Hill, New York, 1940), pp. 323–324; W. G. Pollard, Phys. Rev. **56**, 324 (1939).

[14] Y. Shamir, Nucl. Phys. **B406**, 90 (1993).

[15] C. Callan and J. Harvey, Nucl. Phys. **B250**, 427 (1985).

[16] A. Coste and M. Lüscher, Nucl. Phys. **B323**, 631 (1989).

[17] S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976).

[18] K. Jansen and M. Schmaltz, Phys. Lett. B **296**, 374 (1992).

[19] S. Aoki and H. Hirose, Phys. Rev. D **49**, 2604 (1994).

[20] S. Chandrasekharan, Phys. Rev. D **49**, 1980 (1994).

[21] Y. Shamir, Weizmann Report No. WIS-93-99-PH (hep-lat/9310006), 1993 (unpublished).

[22] M. Golterman, K. Jansen, D. Petcher, and J. Vink, Phys. Rev. D **49**, 1606 (1994).

[23] M. Chu, J. Grandy, S. Huang, and J. Negele, Phys. Rev. D **49**, 6039 (1994).

[24] I. Montvay, in *Lattice '92* [2], p. 621; Phys. Lett. B **199**, 89 (1987).

[25] J. Distler and S. Rey, Princeton Report No. PUPT-1386, 1993 (unpublished).

[26] S. Frolov and A. Slavnov, Nucl. Phys. **B411**, 647 (1994); S. Aoki and Y. Kikukawa, Mod. Phys. Lett. A **8**, 3517 (1993).