# SO(3) scale parameter

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With Monte Carlo methods we calculate the ratio of scale parameters for the adjoint and fundamental versions of SU(2) lattice gauge theory. We find good agreement with the perturbatively predicted value.

The standard SU(3) Yang-Mills theory of quarks interacting through gluon exchange is widely accepted as the fundamental theory for strong interactions. In comparison with the theories for weak electromagnetic interactions, it has yielded a limited range of testable predictions. Perturbation methods and Monte Carlo simulation on a lattice are the two common schemes for extracting predictions. They are complementary: perturbative results apply to short-distance interactions where the coupling is small (approaching asymptotic freedom), while lattice work has been concerned mainly with such long-distance, low-energy effects as the confining interquark potential. Because the two techniques are usually disjoint, points of contact between them have a special significance.

As an ultraviolet cutoff, the lattice removes divergences and permits numerical studies of the theory as formulated directly out of bare parameters. We expect that a gauge theory will remain well defined as the cutoff is removed, although there is no proof yet. (Such a proof would establish the existence of a nontrivial four-dimensional quantum field theory.) If the theory is well defined as we expect, it should yield physical predictions that do not depend on the cutoff scheme. Perturbative predictions have been extensively checked for regularization-scheme independence, but now we can use lattice methods to test scheme independence by varying the details of the lattice cutoff.

Numerous variants of Wilson's formulation[1] have been studied. These include changes in the plaquette action,[2] changes in shapes of the basic loops used to define the theory,[3] and more complicated lattices than the simple hypercubic.[4] The Wilson action is a sum over all plaquettes of the plaquette action

$$S_{\square}=\beta \operatorname{tr} U_{\square} , \tag{1}$$

where $\beta$ is a constant and $U_{\square}$ is the ordered product of group elements around the plaquette. The group elements are in the fundamental representation.

A general plaquette action preserving gauge invariance would be a sum of traces in different representations. For example, the SU(2) mixed plaquette action

$$S_{\square}=\beta(1-\tfrac{1}{2}\operatorname{tr} U_{\square})+\beta_A(1-\tfrac{1}{3}\operatorname{tr}_A U_{\square}) \tag{2}$$

has the first term in the fundamental representation and the second term in the adjoint representation. This action reproduces the classical Yang-Mills action if

$$\frac{1}{{g_0}^2}=\frac{\beta}{4}+\frac{2\beta_A}{3} , \tag{3}$$

where $g_0$ is the bare coupling constant. In this paper we investigate the connections between SU(2) theory in the fundamental ($\beta_A=0$) and adjoint ($\beta=0$) formulations.

The adjoint theory has been the subject of numerous studies. Early work showed that it has a first-order phase transition.[5] In the fundamental theory no phase transition is found, which supports the confinement hypothesis. The adjoint transition was somewhat puzzling because the classical continuum limit of both the fundamental and adjoint actions is the Yang-Mills field theory. Subsequent work with the mixed action showed how to continuously vary either the strong- or weak-coupling phases of the SO(3) model into the SU(2) theory.[6] Thus the adjoint transition is not deconfining. It has been ascribed to a condensation of $Z_2$ monopoles; these monopoles are exponentially suppressed in the weak-coupling regime and should not influence the continuum theory.

This paper concentrates on the asymptotic-freedom scales of the two formulations. They are interesting partly for technical reasons. First, a rather extreme variation is predicted perturbatively: the ratio of the two scales is 28.9, a large number. To verify this ratio is to directly compare perturbative and Monte Carlo techniques, testing the reliability of both.

Second, the adjoint action has a peculiarity, in that it does not distinguish between elements of the SU(2) group differing by factors from the group center. Thus expectation values of Wilson loops in the fundamental representation automatically vanish. Numerical comparisons of the two versions must rely on adjoint loops alone. Monte Carlo matching of variant actions is usually based on the string tension, measured by large fundamental loops. In this way the scale parameters for the Wilson, Manton, and Villian actions have been measured.[7,8] Here we must match some other quantity.[9]

Third, the phase transition in the adjoint model occurs

 

at a small value of the coupling. Since the two theories are to be compared in the weak-coupling region, we are forced to use extremely fine lattices. In our work, the bare coupling $\beta=4/g_0^2$ is in the neighborhood of 10. With previous measurements of the string tension $K$ and the physical SU(3) result $\sqrt{K}=400$ MeV, this represents a lattice spacing of about $10^{-22}$ cm, a small value indeed. The string tension would be far too small to measure. We want to show that even these conditions permit useful Monte Carlo simulation, and our results should be more clear-cut since the former method requires guessing about an appropriate scaling region. Since we are not restricted to this region, we can also minimize corrections of order $g^2$ which contribute when the ratio of scale parameters is measured.[10]

We start with measurements of the ratio $R_2$, where $R_{2N}$ is defined by

$$R_{2N}=W(2N,2N)W(N,N)/[W(2N,N)]^2 \tag{4}$$

and $W(I,J)$ is the expectation value of a Wilson loop of dimensions $I$ times $J$ in lattice units. The loops are measured in the adjoint representation since, as mentioned above, the adjoint theory makes fundamental loops vanish. This ratio is designed to make any perturbative divergences in the loops themselves cancel. $R_2$ has a perturbative expansion in terms of the bare coupling

$$R_2=1-p_1 g_0^2(a)-p_2 g_0^4(a)+\cdots . \tag{5}$$

The constant $p_1$ is independent of the choice of action, but $p_2$ is not. We can define a renormalized coupling, at this scale, by[11]

$$g_r^2=(1-R_2)/p_1 \tag{6}$$

using the $R_2$ measured with adjoint loops. Renormalization analysis predicts that $g_0^2$ and $g_r^2$ are both solutions to the equation

$$a\, dg(a)/da=\beta_0 g^3(a)+\beta_1 g^5(a)+\beta_2 g^7(a)+\cdots , \tag{7}$$

where $a$ is the lattice spacing, $\beta_0=11/24\pi^2$, $\beta_1=17/96\pi^4$, and $\beta_2$ depends on the lattice action. The solution is

$$\Lambda a=\left[1+\frac{\beta_1^2-\beta_2\beta_0}{2\beta_0^3}g^2+\cdots\right]\times e^{-1/2\beta_0 g^2}(\beta_0 g^2)^{-\beta_1/2\beta_0^2} \tag{8}$$

with scale parameters $\Lambda_0$ and $\Lambda_r$, respectively. This is a perturbative result which has also been demonstrated in a Monte Carlo simulation.[11] We do not attempt to verify it here, but we use the scaling relationship in determining $\Lambda$.

Now, we can choose either $g_0(a)$ or $g_r(a)$ as our reference coupling. For a given $g_0(a)$, $g_r(a)$ will depend on the choice of action; the difference will show up in $\Lambda_r$. On the other hand, $g_r(a)$ could be set to some physical value which is supposed to be independent of the choice of action. In that case $g_0$ must be adjusted to the choice of action through $\Lambda_0$. The ratio of scales in one case is the inverse of the ratio in the other. Following other authors,[10,12] we fix $g_r(a)$ and calculate the ratio of scales for the bare coupling. According to Eq. (8), as the lattice spacing becomes small, so do $g_0^2(a)$ and $g_r^2(a)$; we can use the first terms of the perturbation expansion. In terms of the fundamental action

$$1/g_r^2(a)=1/g_{0F}^2(a)-p_{2F}/p_1+O(g_{0F}^2(a)) , \tag{9}$$

while for the adjoint action

$$1/g_r^2(a)=1/g_{0A}^2(a)-p_{2A}/p_1+O(g_{0A}^2(a)) . \tag{10}$$

From Eqs. (8)–(10) it follows that $\Lambda_F/\Lambda_A=\exp[(p_{2A}-p_{2F})/2\beta_0 p_1]$.

The constant $p_1$ requires only a short calculation. In the path integral, $W(I,J)$ is

$$W(I,J)=\mathrm{tr}\langle\prod U\rangle=\frac{\int[dU_\mu(n)]e^{-S}\mathrm{tr}(\prod U)}{\int[dU_\mu(n)]e^{-S}} . \tag{11}$$

Here $[dU_\mu(n)]$ represents integration over all links of the lattice, and $\prod U$ is the ordered product of group elements about the $I$ by $J$ loop. The group elements can also be written $U_\mu(n)=\exp[ig_0 A_\mu(n)]$, where $A_\mu(n)\equiv A_\mu^a(n)\sigma^a/2$ in terms of the Pauli matrices $\sigma^a$. Let us define a source $J_\mu(n)$ to be 1 when $J_\mu(n)$ is on the loop and $\mu=1,2,3$, or 4, $-1$ when $J_\mu(n)$ is on the loop and $\mu=-1,-2,-3$, or $-4$; and zero when $J_\mu(n)$ is off the loop. Then we can rewrite $\prod U$,

$$\prod U=\exp\left[ig_0\sum A_\mu(n)J_\mu(n)+\text{commutator terms}\right] . \tag{12}$$

The commutator terms appear because $A_\mu(n)$ is a matrix, but to order $g_0$ can be neglected.

Expanding the exponential, we have

$$W(I,J)=\mathrm{tr}1-(g_0^2/2)\mathrm{tr}\sum\langle A_\mu(m)A_\nu(n)\rangle \times J_\mu(m)J_\nu(n)+\cdots , \tag{13}$$

where the sum is over all pairs of links $m,n$. To put the propagator in a convenient form, gauge-fixing terms must be added to the action, but they do not affect the tree-level result. In a lattice version of the Feynman gauge the propagator is

$$\langle A^a{}_\mu(m)A^b{}_\mu(n)\rangle=\delta^{ab}\delta_{\mu\nu}\int_{-\pi}^{\pi}\frac{d^4q}{(2\pi)^4}\frac{e^{i(m-n)}}{4D} , \tag{14}$$

where $D$ is

$$D=\sin^2(q_1/2)+\cdots+\sin^2(q_4/2) . \tag{15}$$

The integral can be rewritten

$$\int_{-\pi}^{\pi}\frac{d^4q}{(2\pi)^4}\frac{e^{iqn}}{4D}=\tfrac{1}{2}\int_0^\infty dx\, e^{-4x}I(n_1,x)\cdots I(n_4,x) \equiv Z(n_1,n_2,n_3,n_4) \tag{16}$$

with $I(n_1,x)$ denoting the modified Bessel function of order $n_1$, etc. In this form, it can easily be done on a computer. Finally, we get a sum over $Z$ constants:

TABLE I. Perturbative expansions for the ratio $R_{2N}$. For the continuum, the expansion is $R_{00}=1-0.0661C_2g_0^2+\cdots$ and $C_2=2$ for the adjoint representation.

| $2N$ | Expansion |
|---|---|
| 2 | $1-0.0724C_2g_0^2+\cdots$ |
| 4 | $1-0.0766C_2g_0^2+\cdots$ |
| 6 | $1-0.0713C_2g_0^2+\cdots$ |
| 8 | $1-0.0685C_2g_0^2+\cdots$ |
| 10 | $1-0.0673C_2g_0^2+\cdots$ |

$$W(I,J)=\mathrm{tr}1-(g_0^2/2)C_2 \times\sum Z(m_1-n_1,\ldots,m_4-n_4)S_{mn}\ . \quad (17)$$

Here $C_2$ is the quadratic Casimir operator for the representation and the sum is over all pairs of links $m,n$ on the same side or on opposite sides of the loop; $S_{mn}$ is a sign factor which is negative when the pairs are on opposite sides. For example, a 2 by 2 loop in the adjoint representation is, to order $g_0^2$,

$$W(2,2)=1-g_0^2[Z(0,0,0,0)+Z(1,0,0,0) -Z(2,0,0,0)-Z(2,1,0,0)]\ . \quad (18)$$

In this way we have calculated $R_{2N}$ for $N$ up to 5. Table I shows the values, which are already close to the continuum value.[11]

The difference $(p_{2F}-p_{2A})/p_1$ has been calculated with the background-field method, in the limit of vanishing $a$.[12,13] It is due to the single diagram shown in Fig. 1. The value is $\frac{5}{16}$, so $\Lambda_F/\Lambda_A=28.9$ approximately. This is the number we wish to obtain by Monte Carlo simulation of the lattice theory. Recently, there has also been a calculation to order $g^2(a)$ in the series[14]

$$1/g_{0A}^2(a')=1/g_{0F}^2(a)+2\beta_0\ln(a/a')+c_0 +g_{0F}^2(a)[2\beta_1\ln(a/a')+c_1] +O(g_{0F}^4(a))\ . \quad (19)$$

Thus

$$\Lambda_F/\Lambda_A=(1-c_1g^2/2\beta_0) \times\exp[(1/g_{0A}^2-1/g_{0F}^2)/2\beta_0]\ , \quad (20)$$

where $g^2=g_{0F}^2=g_{0A}^2$ to order $g^2$. Our procedure is to measure $1/g_{0A}^2-1/g_{0F}^2$. Since we do this for nonvanishing $g^2$, we include the correction

$$1-c_1g^2/2\beta_0=1-\delta_{F,A}g^2-c_0\beta_1g^2/2\beta_0^2\ , \quad (21)$$

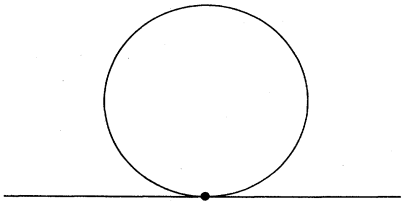

FIG. 1. Gluon self-energy diagram for calculating $(p_{2F}-p_{2A})/p_1$, from Ref. 12.

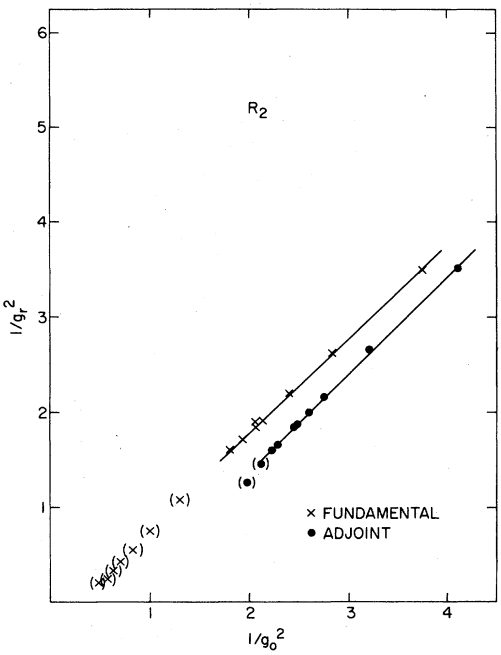


FIG. 2. A plot of $1/g_r^2$ vs $1/g_0^2$, with $1/g_r^2\equiv0.1448/(1-R_2)$. Points shown in parentheses in this and following figures were not used in the analysis.

where

$$\delta_{F,A}=(c_1\beta_0-c_0\beta_1)/2\beta_0^2=1.23$$

is the result of Ref. 11.

The Monte Carlo algorithm has been described before.[15] In this work we used periodic lattices of sizes $6^4$ and $8^4$.

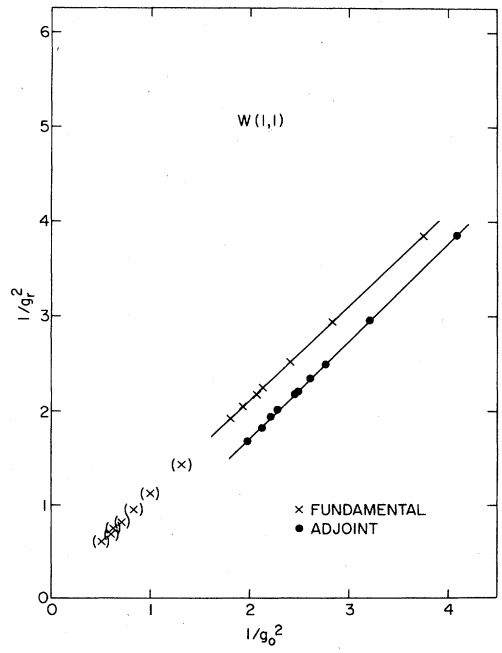


FIG. 3. A plot of $1/g_r^2$ vs $1/g_0^2$, with $1/g_r^2\equiv0.5[1-W(1,1)/3]^{-1}$.

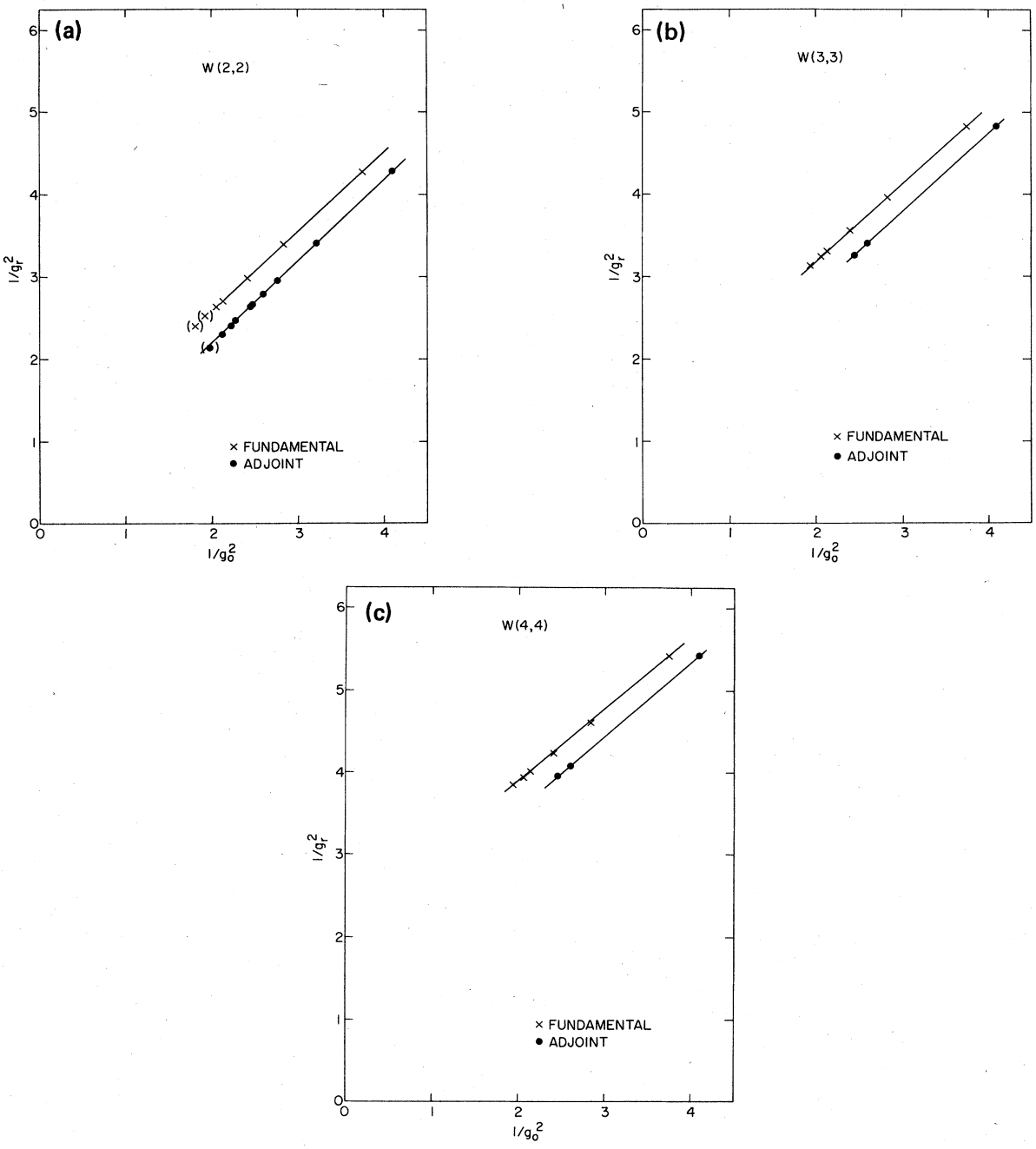


FIG. 4. Plots of $1/g_r^2$ vs $1/g_0^2$, with (a) $1/g_r^2 \equiv 1.37[1-W(2,2)/3]^{-1}$, (b) $1/g_r^2 \equiv 2.31[1-W(3,3)/3]^{-1}$, and (c) $1/g_r^2 \equiv 3.26[1-W(4,4)/3]^{-1}$.

The difference in size did not seem to affect the results. Figure 2 shows the results obtained using $1/g_r^2 \equiv 0.1448/(1-R_2)$. A fit to the fundamental and adjoint points gives slopes to 0.98 and 1.02, respectively. The predicted slope for both, to lowest order, is 1.00. We could not use the results from the $R_4$ ratio because the fluctuations were too high.

Up to this point we have been considering Wilson loop ratios which cancel perimeter and corner divergences. These ratios form physical quantities which have been useful for renormalization-group studies. But here we are matching quantities directly on identical lattices. The divergences of the Wilson loops are purely perturbative, and once the finite difference due to the different actions is taken into account, the divergent parts should cancel. Thus for our purpose of determining the size of this finite difference, we can directly match the adjoint loops themselves. Given any adjoint loop, the difference due to choice of action is given, in leading order, by the diagram shown in Fig. 1. Small loops have reduced statistical errors, so there is an advantage to using the loops directly instead of the loop ratios.

TABLE II. Raw and corrected values of $\Lambda_F/\Lambda_A$ obtained from $R_2$, $W(1,1)$, $W(2,2)$, $W(3,3)$, and $W(4,4)$. The values of $1/g_r^2$ chosen correspond roughly to $1/g_0^2=2$, 3, and 4.

| Data | $1/g_r^2$ | $(\Lambda_F/\Lambda_A)_{\text{raw}}$ | $(\Lambda_F/\Lambda_A)_{\text{corrected}}$ |
|---|---|---|---|
| $R_2$ | 1.59 | 88.1 | 21.2—33.8 |
| $R_2$ | 2.61 | 51.6 | 26.8—29.6 |
| $R_2$ | 3.48 | 32.6 | 20.7—21.6 |
| $W(1,1)$ | 2.00 | 71.4 | 19.9—28.8 |
| $W(1,1)$ | 3.00 | 49.2 | 26.1—28.7 |
| $W(1,1)$ | 4.00 | 34.0 | 22.2—23.1 |
| $W(2,2)$ | 2.46 | 72.4 | 20.0—29.2 |
| $W(2,2)$ | 3.29 | 53.5 | 26.8—30.0 |
| $W(2,2)$ | 4.38 | 35.8 | 23.1—24.1 |
| $W(3,3)$ | 3.00 | 71.3 | 17.3—27.1 |
| $W(3,3)$ | 3.92 | 54.1 | 27.7—30.8 |
| $W(3,3)$ | 4.84 | 41.1 | 26.3—27.5 |
| $W(4,4)$ | 3.59 | 78.6 | 14.0—26.7 |
| $W(4,4)$ | 4.57 | 52.9 | 26.9—30.0 |
| $W(4,4)$ | 5.54 | 35.6 | 23.1—24.1 |

In particular, we can define a new renormalized coupling from the elementary Wilson loop:

$$\frac{1}{g_r^2(a)}=\frac{\frac{1}{2}}{1-W(1,1)/3}\ . \tag{22}$$

This coupling should satisfy the renormalization-group equation with its own scale parameter, which can be matched across the actions. Figure 3 shows the measured values for this coupling as a function of bare fundamental and adjoint couplings. Figure 4 shows the corresponding comparison for larger loops. In each case $1/g_r^2$ is defined as $p_{1N}[1-W(N,N)/3]^{-1}$, where $W(N,N)=1-p_{1N}g_0^2+\cdots$.

In Table II we list results for the ratio $\Lambda_F/\Lambda_A$ both with and without the order-$g^2$ corrections. The range in the latter is due to the range in $g^2$ (since $g_{0F}^2$ and $g_{0A}^2$ are not exactly equal). Without the correction, there is rough agreement, and for smaller $g^2$ the measured values approach the predicted 28.9. Including the correction, the agreement is very good overall.

In summary, we have shown how matching of adjoint loops and their ratios as obtained from Monte Carlo simulation permits a reasonably accurate determination of the relation between SU(2) and SO(3) scale parameters. The results represent a direct comparison of perturbative and Monte Carlo predictions. Further, this work shows the usefulness of Monte Carlo simulation deep in the weak-coupling regime, where standard quantities such as the string tension would be unmeasurably small. Finally, the fact that the numerical results connect so well with the perturbative limit lends support to the existence of the continuum limit of SU(2) lattice gauge theory, and to the independence of the limit from the choice of lattice action.

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