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Phase transitions in U(N) lattice gauge theory in four dimensions

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Monte Carlo methods are used to study pure U(N), $N=2$, 3, 4, 5, and 6, gauge fields in four space-time dimensions. Using Wilson's form of the action, first-order phase transitions are clearly seen at $\beta_c=3.30$, 6.88, 12.14, 18.8, and 27.0 for U(2), U(3), U(4), U(5), and U(6), respectively.

Recent Monte Carlo analyses[1-8] have found unexpected first-order phase transitions in SU(N) lattice gauge theory for N larger than 3. These transitions are probably not deconfining and may be artifacts of the Wilson action. In this paper we study U(N) lattice gauge theory and find a first-order transition already at $N=2$.

Why should we study U(N)? In the continuum field theory, the model trivially reduces to the direct product of decoupled SU(N) and U(1) gauge fields. Our investigations can only provide information on lattice artifacts. These, however, may be useful for an analytic understanding of the confinement mechanism.[9]

On the lattice, U(1) gauge theory has a second-order phase transition[10-12] between strong-coupling confinement and weak coupling with massless photons as spin waves. As free photons should also occur in the weakly coupled U(N) theory, some nontrivial phase structure is expected. However, in the lattice theory the U(1) and SU(N) fields are highly coupled and the nature of the transition may change. Also, the large-N SU(N) transitions may become entwined with the U(1) critical behavior. We will see that this is indeed the case in that all the U(N) models studied here show a single first-order transition.

We study Wilson's lattice gauge theory[13] for the gauge groups U(N), for $N=2$, 3, 4, 5, and 6. We work on a hypercubical lattice of four Euclidean space-time dimensions. On each link $\{i,j\}$ joining nearest neighbors i and j sits a matrix U_{ij} from U(N). We can always write

$$U_{ij}=\exp(i\theta_{ij})\bar{U}_{ij} \ ,$$

where $\bar{U}_{ij}$ is an $N\times N$ unitary unimodular matrix of SU(N). By this device we can separate out the degrees of freedom θ associated with the center of the group. The group manifold is covered when θ sweeps over the interval $[0,2\pi/N]$ and $\bar{U}$ sweeps over SU(N). We define our partition function by

$$Z(\beta)=\int\left[\prod_{\{i,j\}}dU_{ij}\right]\exp(-\beta S[U]) \ ,$$

where the inverse temperature β is related to the bare coupling constant g_0 by $\beta=2N/g_0^2$. The measure in the partition function is the normalized invariant Haar measure for the group. The action[13] S is the sum over all plaquettes $\square$ in the lattice,

$$S[U]=\sum_{\square}S_{\square}=\sum_{\square}\left(1-\frac{1}{N}\operatorname{Re}\operatorname{Tr}U_{\square}\right) \ .$$

Here the sum extends over all elementry squares or plaquettes of the lattice and $U_{\square}$ is an ordered group product of link variables around a plaquette. The system is equilibrated using the method of Metropolis *et al.*[14] Once the system is in a state of statistical equilibrium, we measure the average action per plaquette $\langle E\rangle$.

Using well-known techniques it is easy to establish that, for U(N) and SU(N), the leading-order high-temperature expansions are given by

$$\langle E\rangle=\begin{cases}1-\dfrac{\beta}{2N^2}+O(\beta^3), & \text{(1a)}\\ 1-\dfrac{\beta}{2N^2}(1+\delta_{2,N})-\dfrac{\beta^2}{216}\delta_{3,N}+O(\beta^3) \ , & \text{(1b)}\end{cases}$$

respectively, while the leading-order low-temperature expansions for U(N) and SU(N) are given by

$$\langle E\rangle=\begin{cases}N^2/4\beta+O(\beta^{-2}), & \text{(2a)}\\ (N^2-1)/4\beta+O(\beta^{-2}) \ , & \text{(2b)}\end{cases}$$

respectively. Equations (2) represent $\frac{1}{2}kT=1/2\beta$ per nongauge degree of freedom.

In Figs. 1(a)–1(e) we show the average action per plaquette for U(2), U(3), U(4), U(5), and U(6),

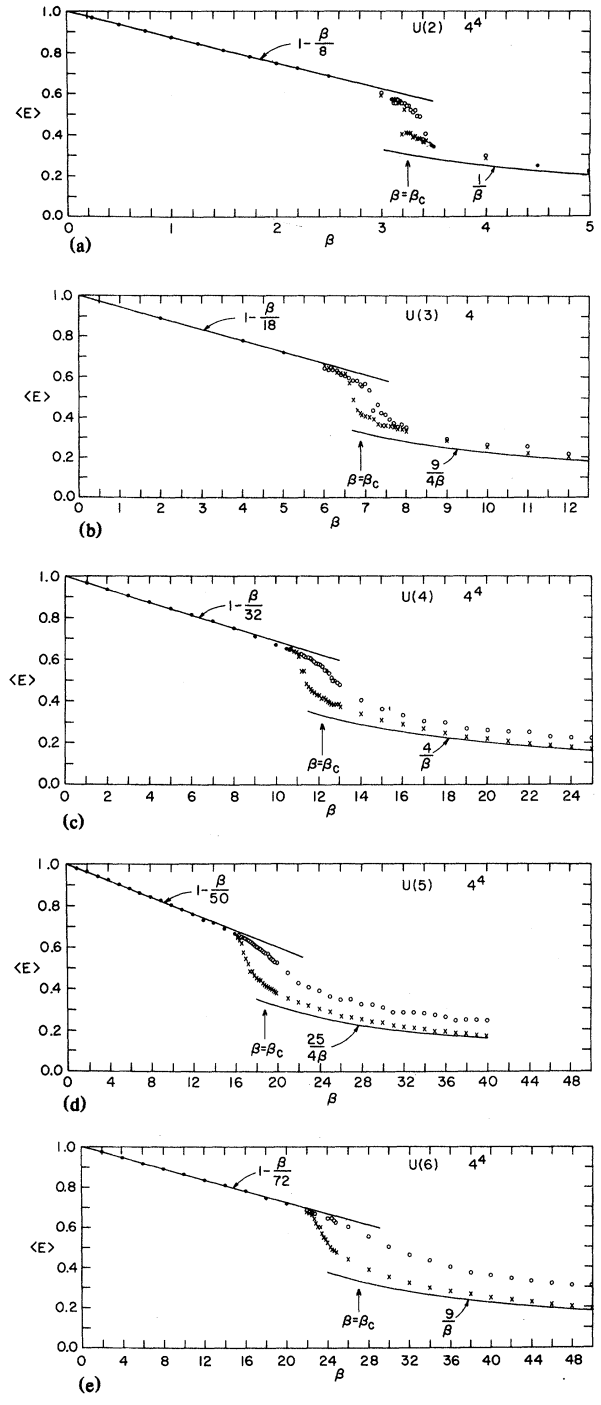

FIG. 1. The average action per plaquette for (a) U(2), (b) U(3), (c) U(4), (d) U(5), and (e) U(6) on a 4^4 lattice as a function of the inverse temperature β. The solid circles were extracted from the evolution of the ordered and disordered starts, while the crosses and open circles represent the average over the last 20 of 100 iterations through the lattice for ordered and disordered starting lattices, respectively. The curves represent the leading-order high- and low-temperature expansions of Eqs. (1a) and (2a), respectively.

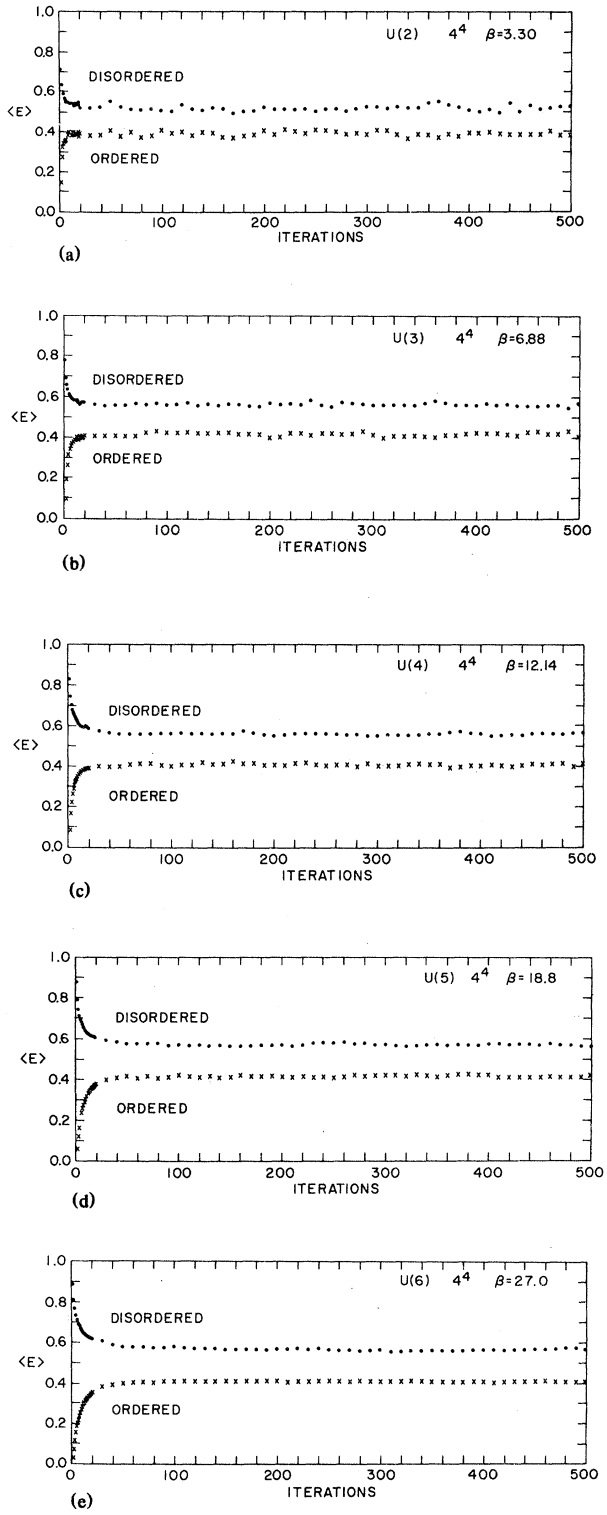

FIG. 2. The evolution of the ordered and disordered configurations for the average action per plaquette for (a) U(2), (b) U(3), (c) U(4), (d) U(5), and (e) U(6) for a 4^4 lattice at the appropriate critical inverse temperatures.

respectively, on a 4^4 lattice as a function of the inverse temperature β. All the data points are the result of 100 iterations through the lattice with an average over the last 20 iterations. In the high-temperature region the average action per plaquette for ordered and disordered starting lattices quickly converged to a unique value. The relaxation time for the lattice lengthens in the crossover region between the high- and low-temperature regimes and hysteresis loops are clearly visible. The critical inverse temperatures are estimated from the hysteresis loops to be $\beta_c = 3.30 \pm 0.05$, 6.88 ± 0.5, 12.14 ± 0.7, 18.8 ± 1.1, and 27.0 ± 2.6 for U(2), U(3), U(4), U(5), and U(6), respectively. Beyond the crossover region in the low-temperature regime the average action per plaquette for ordered and disordered starting lattices converges after 100 iterations to a unique value for U(2) [Fig. 1(a)] and U(3) [Fig. 1(b)]. However, for U(4) [Fig. 1(c)], U(5) [Fig. 1(d)], and U(6) [Fig. 1(e)] more than 100 iterations through the lattice are needed for ordered and disordered starting lattices to converge to a unique value of the average action per plaquette. This may be due to insufficient optimization of the algorithm with these large groups.

Also shown in Fig. 1 are the leading-order high- and low-temperature expansions of Eqs. (1a) and (2a), respectively. For $\beta << \beta_c$ the Monte Carlo-generated data fit the first two terms of the high-temperature expansions almost exactly. For $\beta >> \beta_c$ the data are certainly converging on the leading-order low-temperature expansions and these act as lower bounds on the data. In Figs. 1(a)–1(e), no evidence is found for any structure near the U(1) critical point[10–12] at $\beta \simeq 1.0$.

Figures 2(a)–2(e) show the average action per plaquette as a function of the Monte Carlo iterations for the U(2), U(3), U(4), U(5), and U(6) gauge groups, respectively, on periodic 4^4 hypercubical lattices for both ordered and disordered starts. All of these diagrams correspond to 500 iterations through the lattice at the approximate critical inverse temperatures for their respective gauge groups. The ordered and disordered starting lattices converge to two distinct values of the average action per plaquette, indicating the presence of two distinct phases. We can see that the critical inverse temperatures of U(4), U(5), and U(6) are near those for SU(4),[5,6] SU(5),[5,7] and SU(6),[8] respectively, suggesting the $U(N) \rightarrow SU(N)$ for large N.

In Fig. 3 we show the average action per plaquette as a function of the Monte Carlo iterations for the U(2) gauge theory on a periodic 4^4 hypercubical lattice for both ordered and disordered starts for two

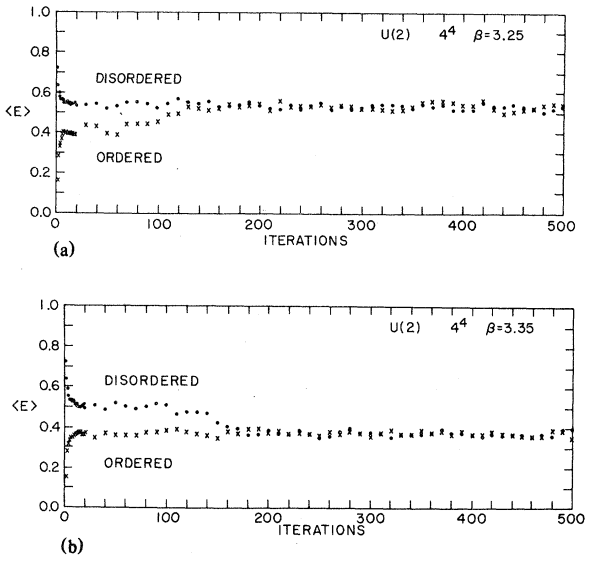

FIG. 3. The evolution of the ordered and disordered configurations for the average action per plaquette for U(2) for a 4^4 lattice at two values of the inverse temperature (a) $\beta = 3.25$ and (b) $\beta = 3.35$.

values of the inverse temperature on either side of the critical inverse temperature. For $\beta = 3.25$ [Fig. 3(a)] and $\beta = 3.35$ [Fig. 3(b)] the ordered and disordered starting lattices converge to a unique value of the average action per plaquette after approximately 110 and 160 iterations, respectively. This is to be compared with Fig. 2(a) where we saw that after 500 iterations the ordered and disordered starts failed to converge to a unique value. This is fairly convincing evidence that we have a first-order phase transition in U(2) gauge theory. A similar situation obtains for the U(3), U(4), U(5), and U(6) gauge groups.

Recently, a technique[15] has been developed for locating the critical point in $U(N)$ gauge theories on the lattice with the predicted result that $\beta_c = 3.16$, 7.11, 12.64, 19.75, and 28.44 for U(2), U(3), U(4), U(5), and U(6), respectively. We can see that these results are in good agreement with our own.

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