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CONFINEMENT AND LATTICE GAUGE THEORY†

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ABSTRACT

I review the motivation for formulating gauge theories on a lattice to study non-perturbative phenomena. I discuss recent progress supporting the compatibility of asymptotic freedom and quark confinement in the standard SU(3) Yang-Mills theory of the strong interaction.

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Lattice gauge theories currently provide one of the most promising approaches towards a demonstration of quark confinement through their interactions with non-Abelian gauge fields. I will first discuss the motivations for the lattice method of defining a gauge theory. Then I will turn to the recent developments which strongly support the coexistence of confinement and asymptotic freedom in the continuum limit of the standard SU(3) gauge model of the strong interaction.

In four-dimensional space time, confinement must be a non-perturbative phenomenon. Suppose quarks are confined by a linear potential

$$E(r) \to Kr \tag{1}$$

where $E(r)$ is the interaction energy of a quark-antiquark pair separated by distance r, and K is a constant referred to as the "string tension" by analogy with the string model. As K is a physical quantity, it must obey the renormalization group equation. This implies the dimensional transmutation[1] form at weak coupling

$$K \sim \frac{1}{a^2} \exp\left(-\frac{1}{\beta_0 g_0^{\,2}(a)}\right) \tag{2}$$

when g_0 is the bare coupling constant defined when an ultraviolet cutoff of length a is introduced. Here β_0 is the first term in a perturbative expansion of the Gell-Mann Low function[2]

$$a \frac{\partial}{\partial a} g_0 = \beta_0 g_0^{\,3} + \beta_1 g_0^{\,5} + O(g_0^{\,7}) \tag{3}$$

The important consequence of Eq. (2) is the essential singularity at $g_0^{\,2}=0$ which precludes any perturbative calculation of K.

Any true non-perturbative analysis of a Yang-Mills theory requires a means of controlling ultraviolet divergences in a manner independent of Feynman diagrams. This precludes most conventional regularization schemes. Herein lies the main virtue of a lattice formulation. As wavelengths less than twice the lattice spacing, a, have no meaning, the discretized theory has a cutoff of each momentum component at π/a.

As with any cutoff prescription, considerable freedom remains in a lattice formulation. Upon removal of the cutoff, the physics of a renormalizable field theory should be independent of the details of the regulator. However, with the cutoff in place, one is free to add to the Lagrangian terms which will not contribute in the continuum limit. Using this freedom, Wilson has presented a particularly elegant lattice formulation for gauge theories.(3) His prescription keeps local gauge invariance as an exact symmetry in a mathematically well-defined system.

Wilson uses the concept that a gauge field represents a theory of non-integrable phase factors.(4) When a material particle transverses a world line C in space time its interaction with a gauge field appears as a phase factor in its wave function

$$\psi \rightarrow \exp(ie \int_C A_\mu dx_\mu) \tag{4}$$

The dynamics of the material particle follow from a path integral over all possible world lines. For a non-Abelian theory these phase factors became matrices in the gauge group. These group elements define the basic degrees of freedom in the lattice theory. Approximating a world line for the particle by a sequence of steps between neighboring sites in a hypercubical lattice, Wilson replaces the above phase factor with the product of phases associated with each step in this path. This leads to using as variables an element U_{ij} of the gauge group associated with every nearest neighbor pair of sites {i,j} in the lattice. For the strong interaction model, these elements are of the group SU(3).

Using these variables, we need an expression for the action which reduces in the continuum limit to the ordinary gauge theory action

$$S(U) \xrightarrow[a\to 0]{} \int d^4x \, \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{4}$$

Wilson proposed the simple form

$$S(U) = \sum_{\Box} \left(1 - \frac{1}{N} \mathrm{Re}(\mathrm{Tr}(\prod_{\{i,j\} \varepsilon \Box} U_{uj}))\right) \tag{5}$$

Here the sum is over all elementary squares or "plaquettes" of the lattice, and the product is an ordered group product of the group elements surrounding the given square. The normalization factor N is chosen for convenience to be the dimension of the group matrices, i.e. 3 for SU(3). A path integral defines the quantum theory

$$Z = \int \prod_{\{i,j\}} dU_{ij} \; e^{-\beta S(U)} \tag{6}$$

where β is proportional to the inverse bare coupling constant

$$\beta = \frac{2N}{g_0^2} \qquad \text{for } SU(N) \tag{7}$$

From Eq. (6) is is apparent that the path integral is equivalent to a partition function for the statistical mechanics of a four-dimensional system of spins belonging to the gauge group. Their interaction is through the four spin coupling in Eq. (5). For $\beta << 1$, Wilson has derived a strong coupling series by expanding the exponent in Eq. (6). In this "high temperature" limit confinement is automatic in the sense that the model reduces to a theory of quarks connected by strings with a finite energy per unit length. On the other hand, at weak coupling $\beta >> 1$ a spin wave expansion reproduces conventional Feynman perturbation theory. This series is known to be at best asymptotic, but its existence suggests a possible low temperature phase of free quarks and massless gluons. As this is qualitatively different from the strong coupling regime, one expects at least one phase transition separating these domains if the free quark phase exists at all. Balian, Drouffe, and Itzykson have presented arguments that in enough space-time dimensions such a transition will occur.(5)

Ultimately we are interested in the continuum limit of the theory. This requires taking the bare coupling constant to a critical value so that correlation scales, i.e. inverse physical masses, become large relative to the cutoff represented by the lattice spacing. The perturbative renormalization group indicates one such critical point at vanishing bare coupling.(6) A continuum limit at this point yields the phenomenon of asymptotic freedom; the effective renormalized coupling will go to zero when defined on decreasing length scales. The virtue of this phenomenon is the perturbative prediction of scaling phenomena in high momentum transfer processes.

To have asymptotic freedom in the same phase that Wilson's expansion exhibits confinement, four dimensional space-time should be inadequate to support a transition to a free massless spin-wave phase at any finite coupling. Based on the approximate recursion relation analysis of Migdal and Kadanoff(7), current lore is that four space time dimensions represent a critical case for gauge theories. In more than four dimensions all gauge groups should exhibit a spin-wave phase whereas in less than four dimensions all continuous gauge groups exhibit only strong coupling behavior. In exactly four dimensions only abelian continuous groups should exhibit a non-trivial phase structure; indeed, this is necessary if Wilson's formalism is to describe quantum electrodynamics, the prototype of all gauge theories.

Recently from two rather different techniques, strong evidence has appeared that this "standard" picture of the phase structure of lattice gauge theories is indeed correct. The first technique involves use of Pade techniques to extrapolate from the strong coupling expansion into a regime where weak coupling predictions become valid[(8),(9)] Assuming the linear interquark potential survives the continuum limit, we can use the string tension K to define a renormalization prescription. The bare coupling constant dependence on cutoff follows by holding this tension fixed by adjusting g_0 as the lattice spacing is reduced. One then tests this assumption in the weak coupling limit by comparing with the asymptotic freedom prediction

$$a^2K \rightarrow \frac{K}{\Lambda_0^2}\left(\beta_0 g_0^2\right)^{\left(-\frac{\beta_1}{\beta_0^2}\right)} \exp\left(-\frac{1}{\beta_0 g_0^2}\right) \qquad (8)$$

This is Eq. (2) written to take account of the second coefficient of the Gell-Mann Low function.[(10)] The parameter Λ_0 is a scale relating the weak coupling behavior at short distances to the strong linear potential at long distances. Kogut, Pearson, and Shigemitsu[(8)] have used a Padé extrapolation of the strong coupling expansion for the left hand side of Eq. (8) and found it matches smoothly onto the asymptotic freedom prediction. Using a Hamiltonian variation of Wilson's path integral formulation, Kogut and Shigemitsu find for SU(3)[9]

$$\Lambda_0 = \frac{1}{205}\sqrt{K} \tag{9}$$

The other technique giving recent results on the lattice theory is Monte Carlo simulation. My own recent research has centered here.[11] Considering the path integral of Eq. (6) as a partition function for a statistical system at inverse temperature , the Monte Carlo procedure generates a few configurations which are typical of that system in thermal equilibrium. This is done by making "random" changes of the various group elements in such a manner that asymptotically the probability of any configuration C is proportional to the Boltzmann factor

$$P(C) \sim e^{-\beta S(C)} \tag{10}$$

A simple intuitive algorithm which I have used for SU(2) is to pass through the lattice and successively replace each group element U with a new one U' chosen randomly from the group with weighting

$$dP(U') = dU e^{-\beta S(U')} \tag{11}$$

where the action is calculated with the neighboring variables at their current values. This technique is equivalent to touching a heat bath at inverse temperature β to the links in question. For SU(3), due to the complexity of the group, I am using a less intuitive but computationally simpler algorithm similar to that used by Wilson.[12] In the following, one iteration means one Monte Carlo operation on every link of the lattice.

As the entire lattice is stored in the computer memory, once it is in equilibrium one can measure any desired correlation function. In a gauge theory the natural correlation functions are Wilson loops.[3] Given a closed contour C of links in the lattice, the associated Wilson loop is the expectation of the product of the link variables

$$W(C) = \left\langle \frac{1}{2} \mathrm{Tr}\left(\prod_{ij \in C} U_{ij} \right) \right\rangle \tag{12}$$

when the U's are ordered sequentially around the contour. If confinement occurs with a linear potential, then W(C) should fall exponentially with the area enclosed by the contour C when the loop becomes large. The coefficient of this falloff is given by the string tension

$$W(C) \sim \exp(-a^2 K\, N_\square(C)) \tag{13}$$

where $N_\square(C)$ is the minimum number of elementary squares covering a surface enclosed by C. In my Monte Carlo treatment, I attempt to measure the Wilson loops and extract this area law behavior at various couplings for comparison with the asymptotic freedom prediction in Eq. (8).

In Figure (1) I illustrate the typical convergence of the Monte Carlo procedure for SU(2) gauge theory. Working at $\beta = 2.3$, I plot the average plaquette or internal energy

$$P = \left\langle 1 - \frac{1}{2} \mathrm{Tr}\left(\prod_{ij \in \square} U_{ij} \right) \right\rangle \tag{14}$$

as a function of the number of iterations for a total of thirty iterations. Both ordered ($U_{ij}=1$) and disordered (U_{ij} random) initial conditions are shown with lattices of from 4^4 to 10^4 sites. Note that the convergence rate is essentially independent of lattice size; only the fluctuations grow on the smaller lattices. This supports the absence of a phase transition in this region. This value of β represents the region of slowest convergence for the Monte Carlo procedure for SU(2).

In Fig. (2) I show the evolution from an ordered state on an 8^4 lattice for several values of β. Note that the convergence rate is not strongly β dependent; a slight decrease occurs in the range β = 2.0-2.4. At all β equilibrium is essentially complete after 20 iterations. Convergence is extremely rapid both at high and low temperatures; consequently, the method is not tied to either strong or weak coupling.

In contrast to the non-Abelian case, Fig. (3) shows the evolution from random and ordered initial states of a U(1) gauge theory on a 6^4 lattice. The value of β is chosen to lie at our Monte Carlo estimate of a critical point. Note that convergence is both slower and accompanied by substantial fluctuations. This represents the transition rendering the strong coupling expansion inapplicable to quantum electrodynamics.

Fig. (4) shows the results of thermal cycling several of the models. By slowly increasing the temperature from very cold to very hot and then reducing it again, a region of slow convergence appears as a hysteresis effect. In the figures we show results on SU(2) gauge theory in four and five dimensions as well as the four dimensional U(1) theory. I show SU(2) in five dimensions in order to illustrate that confinement is indeed lost as expected if four dimensions is critical. The transitions in both the four dimensional U(1) and five dimensional SU(2) theories are clear whereas the four dimensional SU(2) model appears much smoother. In order to provide more support for the lack of a transition in the latter model, I now turn to a study of Wilson loops.

In Fig. (5) I show the expectation values of square Wilson loops at $\beta = 3$ as a function of lattice size. These loops lie in a fundamental lattice plane. Each measurement is an average over all similar loops in the lattice and the error bars represent the root mean square fluctuation over five iterations after attaining equilibrium. As intuitively expected, larger loops show the finite lattice effects most strongly.

To extract an area law, I have constructed the quantities

$$\chi(I,J) = -\ln\left(\frac{W(I,J)\ W(I-1,J-1)}{W(I,J-1)\ W(I-1,J)}\right). \tag{15}$$

The motivation for this construction is that overall constants and perimeter behavior in the loops will cancel out. Whenever the loops are dominated by the area law, i.e. I, J >> 1 or at strong coupling,

$$\chi(I,J) \to a^2K \tag{16}$$

However, at short distances and weak coupling, χ should have a perturbative expansion radically different from the essential singularity expected for a^2K. We are thus led to the conclusion that the value of a^2K as a function of coupling is given by the envelope of curves of $\chi(I,J)$ for all I and J.

In Fig. (6) I show the values of $\chi(I,I)$ for I = 1 to 4 plotted versus $1/g_0^2$ for the gauge group SU(2) on a 10^4 lattice. The error bars are the standard deviation of the mean taken from an ensemble of five configurations. At stronger couplings the larger loops have large relative errors but are all consistent with χ approaching the values from smaller loops. On the graph I plot the strong coupling limit

$$X(I,J) = \ln(g_0^2) + O\left(\frac{1}{g_0^4}\right) \tag{17}$$

as well as the weak coupling behavior for a^2K from Eq. (8) with the parameter

$$\Lambda_0 = (1.3\pm.2) \times 10^{-2}\sqrt{K} \tag{18}$$

The error is a subjective estimate.

Fig. (7) is the same as Fig. (6) except here the gauge group is SU(3). Most of the points are from a 4^4 lattice and hence only X(1,1) and X(2,2) are plotted. At $g_0^{-2} = 1.11$ and 1.80 runs on a 6^4 lattice gave X(3,3) as well. The strong coupling behavior is now

$$X(I,J) = \ln(3g_0^2) + O\left(\frac{1}{g_0^2}\right) \tag{19}$$

The plotted weak coupling behavior for a^2K corresponds to

$$\Lambda_0 = (5.0\pm1.5) \times 10^{-3}\sqrt{K} \tag{20}$$

Note the remarkable agreement with the series result in Eq. (9). This may be somewhat fortuitous as the Hamiltonian and Lagrangian formulations need not give the same Λ_0.

At first sight the small numbers in Eqs. (18) and (20) are rather surprising, coming as they do from a theory with no small dimensionless parameter. However, the value of a renormalization scale is in general dependent on renormalization scheme. Since Λ_0 is defined in a weak coupling limit, perturbative calculations to one loop order can relate different definitions.[13] Hasenfratz and Hasenfratz[14] have recently done a lengthy analysis relating the lattice Λ_0 to a more conventional Λ^{MOM} defined by the three point vertex in Feynman gauge at a given scale in momentum space. Their results are

$$\Lambda^{MOM} = 57.5\ \Lambda_0 \qquad SU(2) \tag{21}$$

$$\Lambda^{MOM} = 83.5\ \Lambda_0 \qquad SU(3) \tag{22}$$

Combining the Monte Carlo results with these numbers gives

$$\Lambda^{MOM} = (.75\pm.12)\sqrt{K} \qquad SU(2) \tag{23}$$

$$\Lambda^{MOM} = (.42\pm.13)\sqrt{K} \qquad SU(3) \tag{24}$$

If we accept the string model[15] connection between K and the Regge slope α'

$$K = \frac{1}{2\pi\alpha'} \tag{25}$$

and use $\alpha' = 1.0\ (\mathrm{GeV})^{-2}$, then we conclude for SU(3)

$$\Lambda^{MOM} = 170\pm50\ \mathrm{MeV}. \tag{26}$$

Some caution is necessary in the phenomenological interpretation of Eq. (26) because virtual quark loops have not been included in the calculation.

In conclusion, recent advances in lattice gauge theory have given evidence for the onset of asymptotic freedom using a renormalization prescription based on confinement with a linearly rising long distance potential. In this way, ties are strengthened between the lattice formulation and more conventional perturbative approaches to gauge theory. The calculation of the parameter Λ_0 in terms of the string tension relates the behavior of the theory in long and short distance regimes.

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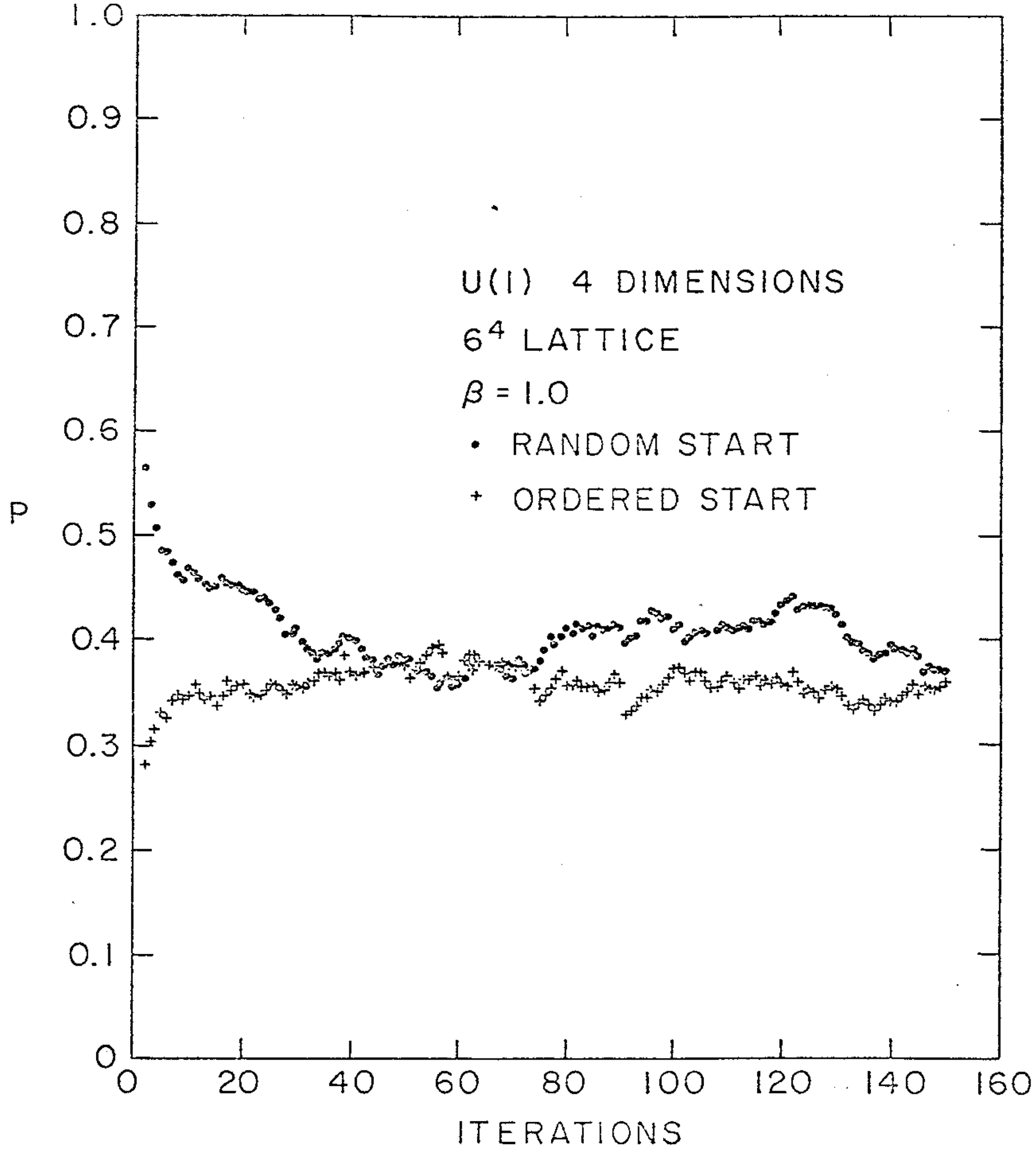


U(1) 4 DIMENSIONS
6^4 LATTICE
β = 1.0
• RANDOM START
+ ORDERED START
P
ITERATIONS
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100
120
140
160

Fig. 3

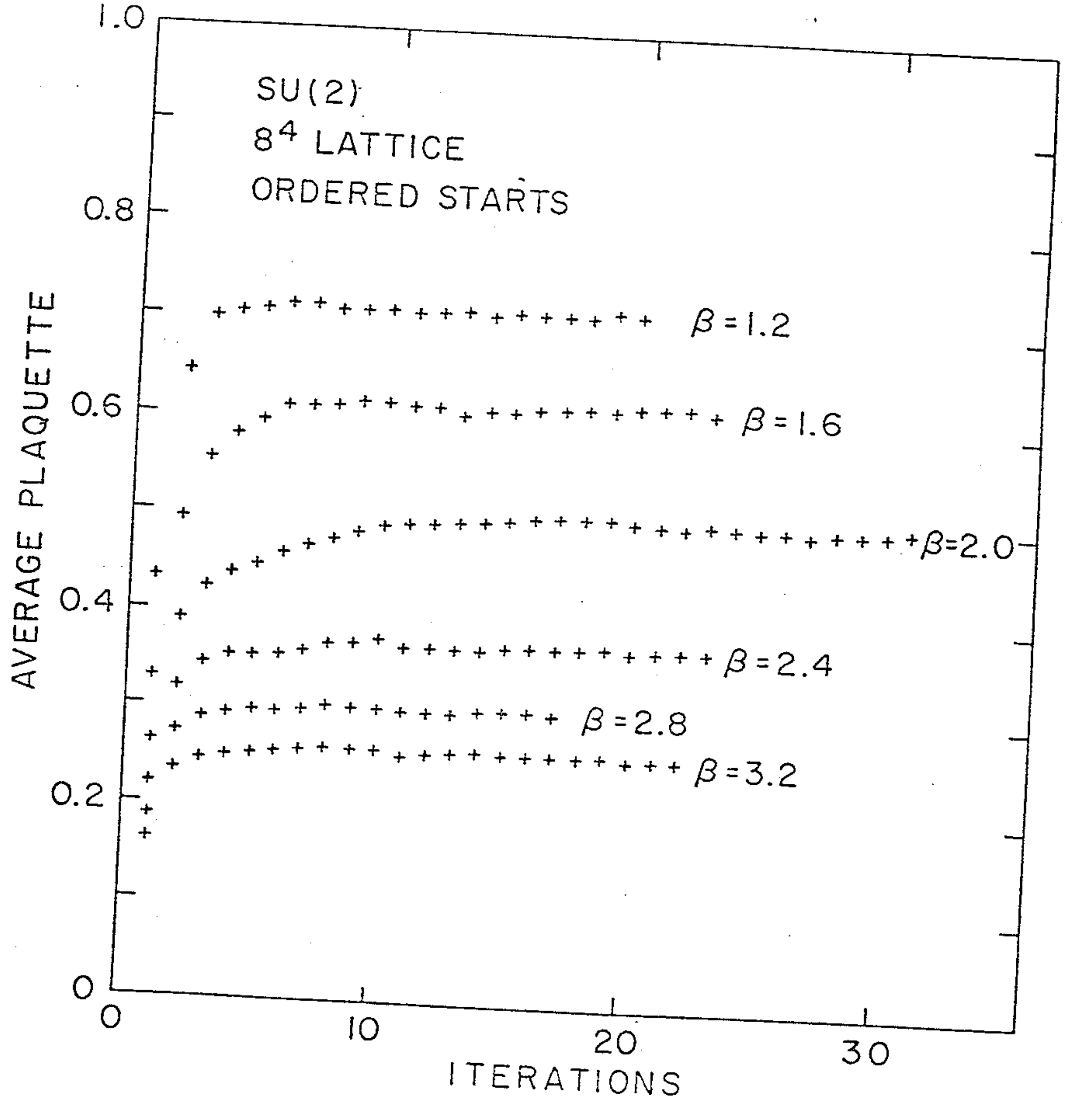

1.0
0.8
0.6
0.4
0.2
0
AVERAGE PLAQUETTE
SU(2)
8⁴ LATTICE
ORDERED STARTS
β=1.2
β=1.6
β=2.0
β=2.4
β=2.8
β=3.2
0
10
20
30
ITERATIONS

Fig. 2

Figure Captions

1. The average plaquette as a function of number of iterations at $\beta = 2.3$ and with the gauge group SU(2).

2. The evolution of the average plaquette at several temperatures.

3. The evolution of the average plaquette for the U(1) gauge group at $\beta = 1.0$.

4. Thermal cycles on (a) the five dimensional SU(2) theory, (b) the four dimensional SU(2) theory, and (c) the four dimensional U(1) (isomorphic to SO(2)) theory.

5. Wilson loops at $\beta = 3$ as a function of lattice size.

6. The quantities X(I,I) for SU(2) gauge theory as a function of inverse coupling squared.

7. The quantities X(I,I) for SU(3) gauge theory.

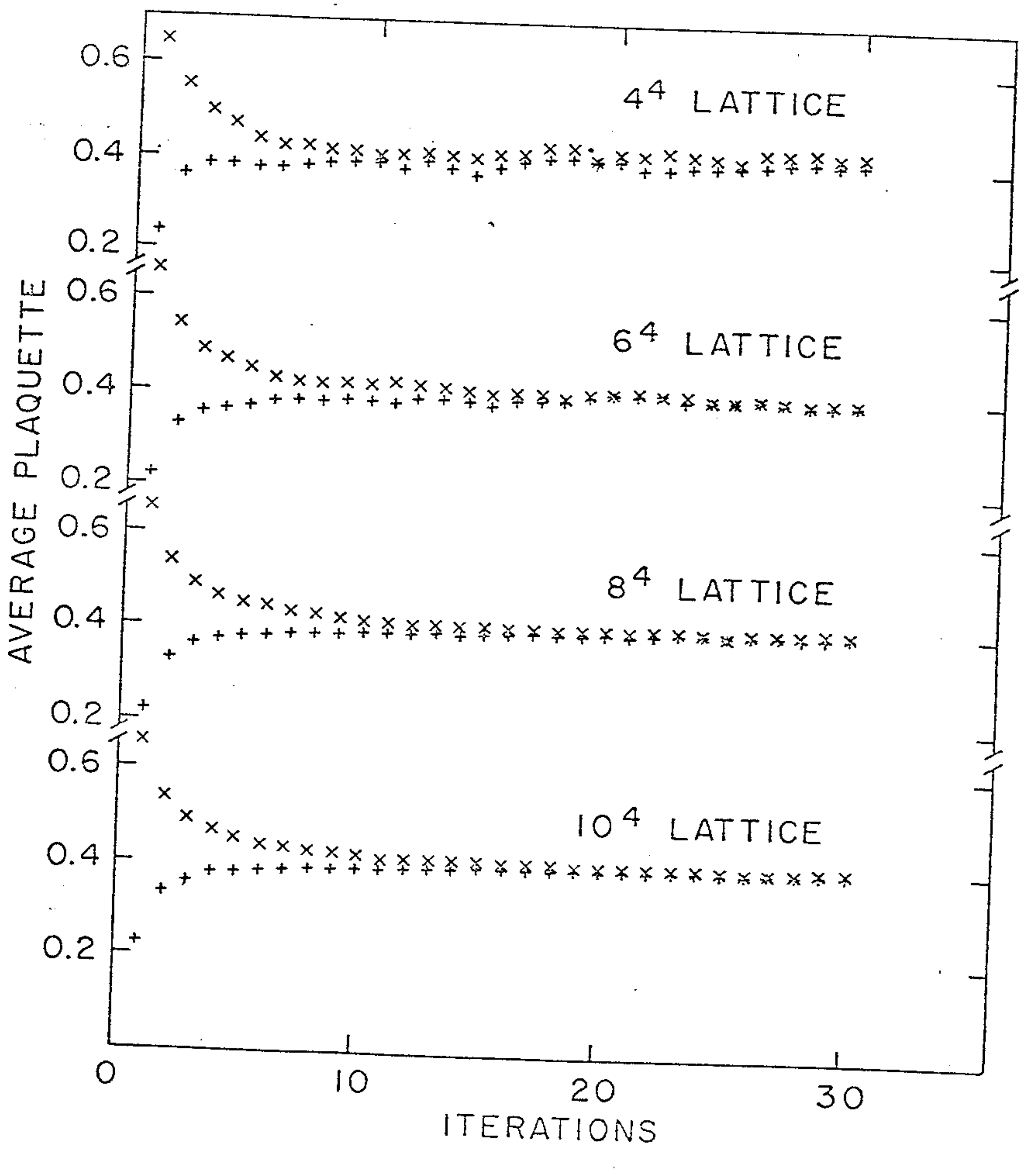
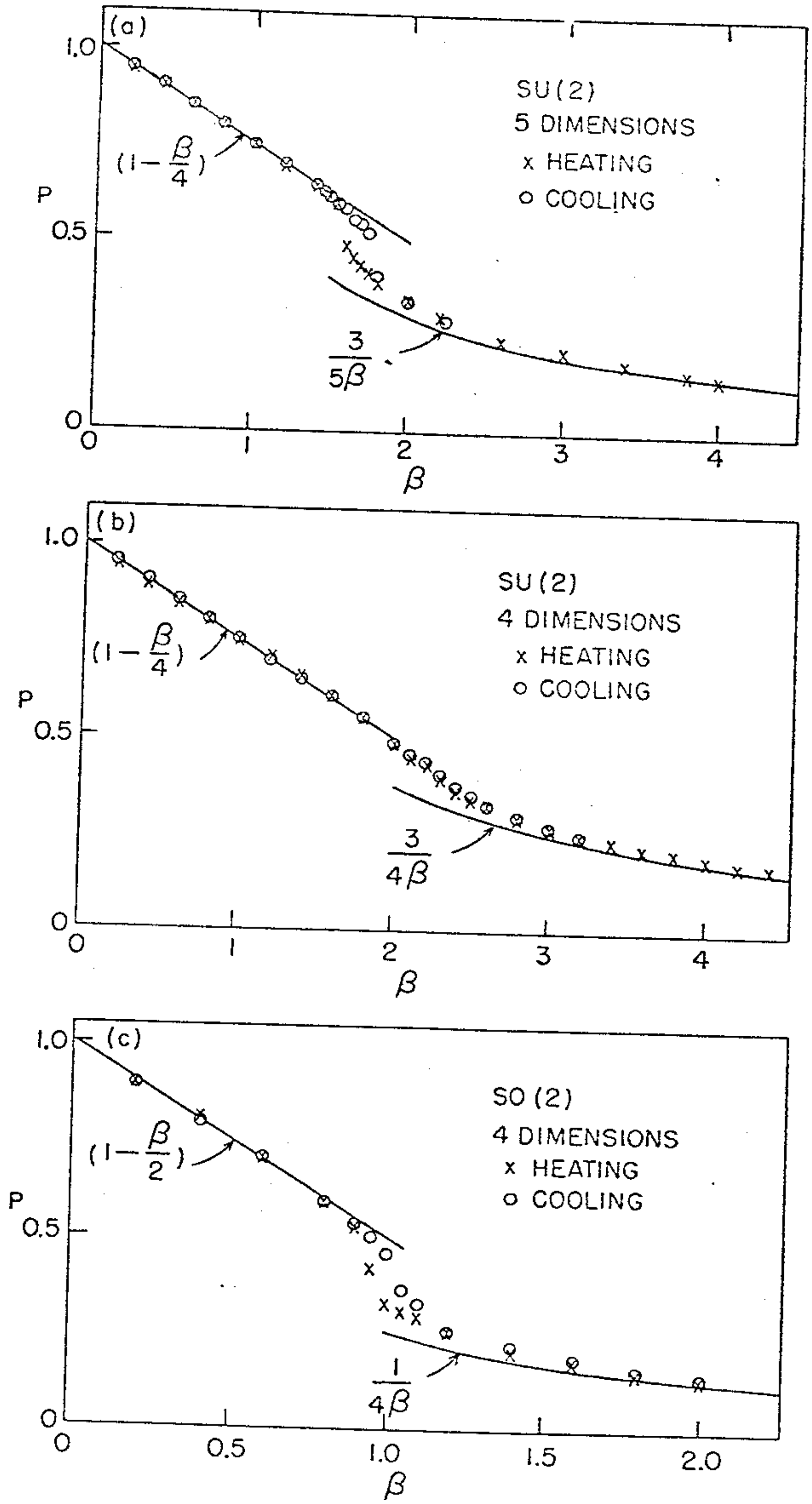
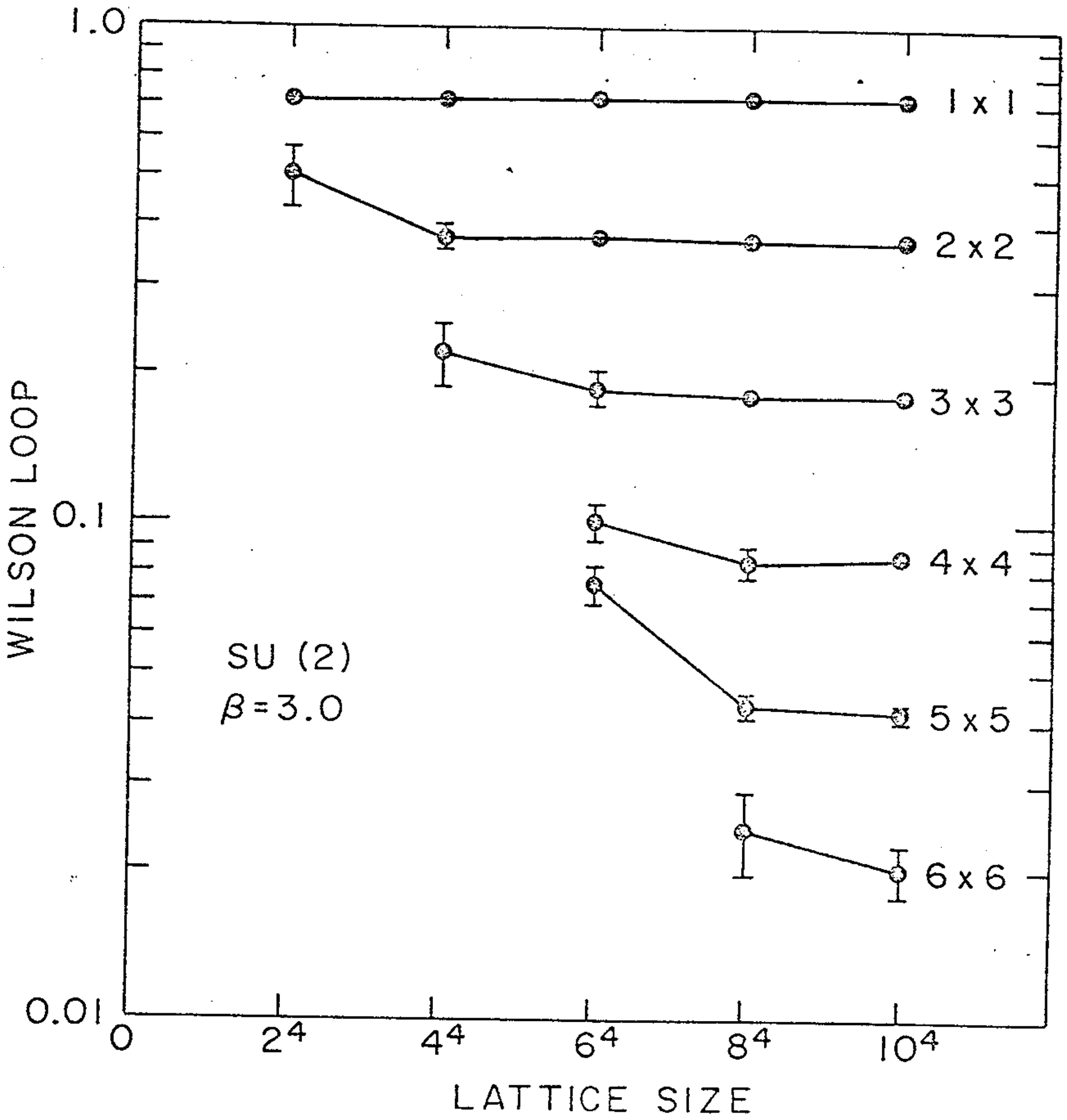

AVERAGE PLAQUETTE
0.6
0.4
0.2
4⁴ LATTICE
6⁴ LATTICE
8⁴ LATTICE
10⁴ LATTICE
0
10
20
30
ITERATIONS

Fig. 1



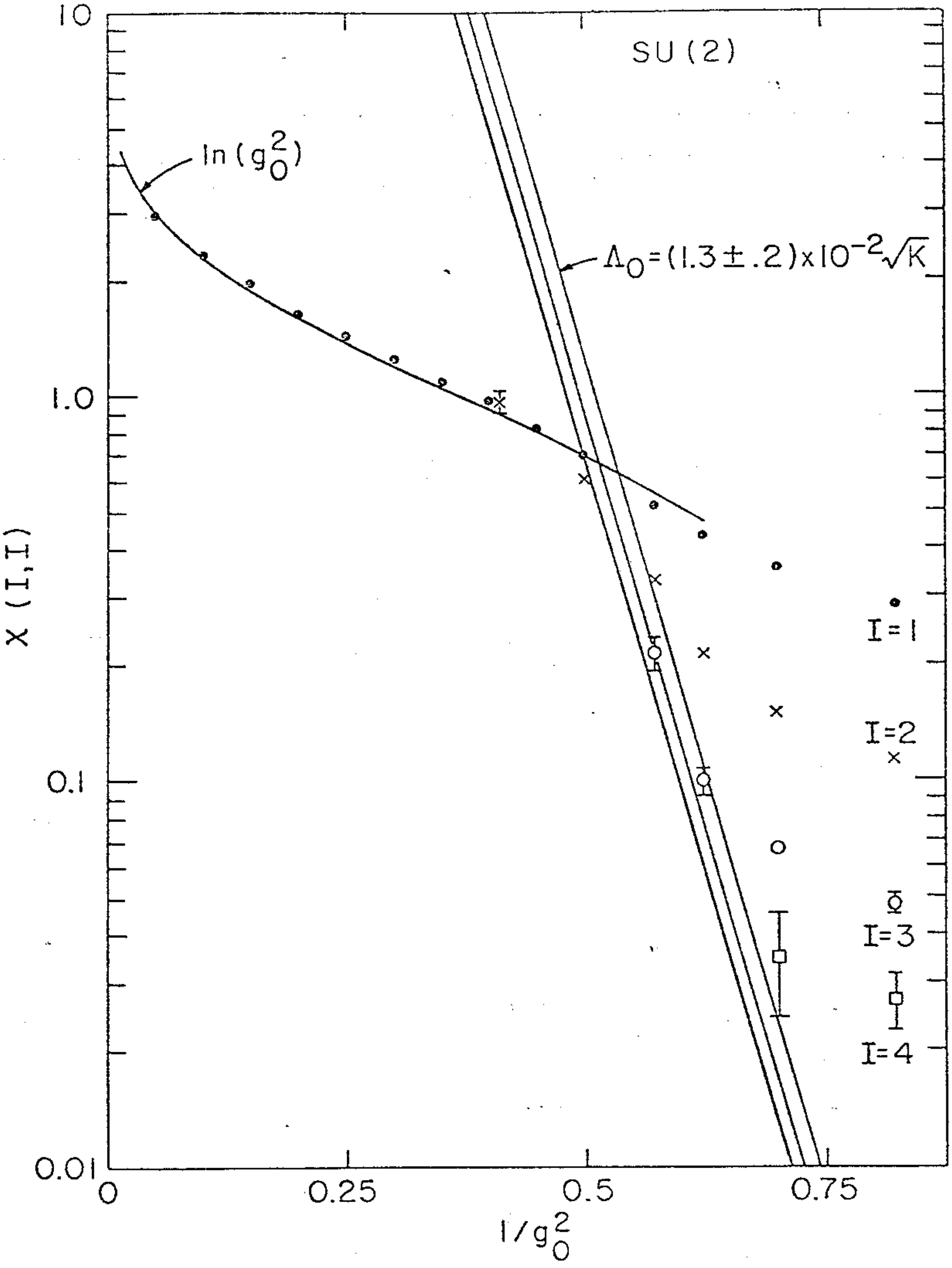
(a)
1.0
0.5
0
P
(1− β/4)
3/5β
SU(2)
5 DIMENSIONS
x HEATING
o COOLING
0
1
2
3
4
β
(b)
1.0
0.5
0
P
(1− β/4)
3/4β
SU(2)
4 DIMENSIONS
x HEATING
o COOLING
0
1
2
3
4
β
(c)
1.0
0.5
0
P
(1− β/2)
1/4β
SO(2)
4 DIMENSIONS
x HEATING
o COOLING
0
0.5
1.0
1.5
2.0
β

Fig. 4



1.0
0.1
0.01
WILSON LOOP
0
2⁴
4⁴
6⁴
8⁴
10⁴
LATTICE SIZE
1 x 1
2 x 2
3 x 3
4 x 4
5 x 5
6 x 6
SU (2)
β = 3.0

Fig. 5



SU(2)
ln(g$_0^2$)
Λ_0=(1.3±.2)x10$^{-2}\sqrt{K}$
X(I,I)
I=1
I=2
I=3
I=4
10
1.0
0.1
0.01
0
0.25
0.5
0.75
1/g$_0^2$

Fig. 6

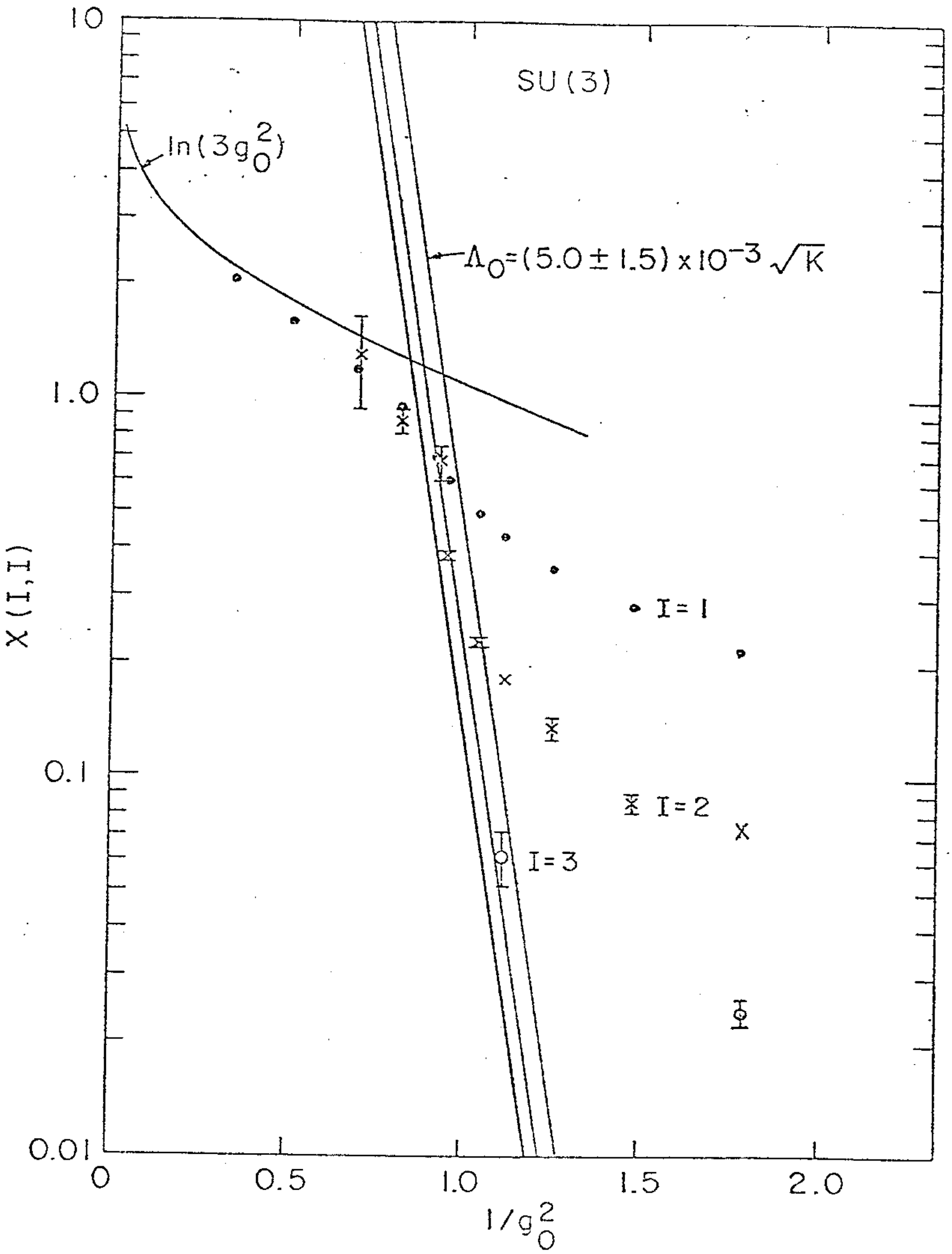

SU(3)
ln(3g$_0^2$)
Λ_0=(5.0 ± 1.5) x 10^{-3} $\sqrt{K}$
I = 1
I = 2
I = 3
X (I, I)
1/g$_0^2$
10
1.0
0.1
0.01
0
0.5
1.0
1.5
2.0

Fig. 7