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Quantum Electrodynamics in the Temporal Gauge

MICHAEL CREUTZ

Physics Department, Brookhaven National Laboratory, Upton, New York 11973

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We canonically quantize electrodynamics in the temporal gauge $A_0 = 0$. Realizing commutation relations in a Hilbert space containing unphysical longitudinal photons, we pay special attention to the implementation of Gauss's law and the attendant normalization difficulties for physical states. We then formulate the perturbation series and explicitly exhibit equivalence with the standard textbook treatment of the Coulomb gauge.

1. INTRODUCTION

Gauge theories dominate the thrust of current thought in theoretical particle physics. Indeed, the popular espousal is that all interactions arise through an underlying gauge structure. The distinguishing feature of a gauge theory is the inability to express the interaction Hamiltonian density in terms of fields with simple transformation properties under the Lorentz group [1]. One necessarily introduces potentials which undergo gauge transformation upon change of frame. Physical observables are formed from gauge invariant combinations of these fields.

The appearance of potentials provides problems for quantization because gauge transformations do not represent physical degrees of freedom. In a path integral approach, Fadde'ev and Popov [2] have shown how to implement a gauge choice so that resulting Green's functions for observables are independent of the particular choice made. A demonstration of equivalence with canonical quantization in a particular gauge then justifies their anzatz [3]. Nevertheless, from a purely canonical point of view the equivalence of various gauges is rather obscure. Indeed, quantum spaces of varying character appear in different treatments. For example, an indefinite metric space is often used for covariant gauges, while all degrees of freedom are physical in the Coulomb gauge.

Recent analyses of nonperturbative phenomena in non-Abelian gauge theories have exposed peculiar complexities of certain gauges. Specification of the Coulomb gauge is not unique for large non-Abelian fields [4]. The axial gauge, where a spacelike component of the potential is constrained to vanish, requires careful handling of surface variables at spatial infinity [5] in order to properly interpret tunneling effects associated with the gauge field topology [6]. The one gauge that has stood out in its simplicity for interpreting these phenomena is the temporal gauge, where the time component of the potential vanishes.

The temporal gauge carries a canonical Hilbert space larger than the set of states of

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physical interest. The extra states are associated with the freedom to perform time-independent gauge transformations. Such transformations do not affect the gauge condition and represent unphysical degrees of freedom that must be eliminated. Implementation of Gauss's law, the generator of such transformations, as a constraint on physical states accomplishes this [7–10]. When massless gauge particles appear in the physical spectrum of the theory, one may restrict attention to those time-independent gauge transformations that vanish at spatial infinity. It is then possible to treat symmetry under the remaining gauge transformations as spontaneously broken, in which case the massless gauge bosons represent Goldstone bosons associated with the broken symmetry [9]. When massless particles do not appear, as in the Higgs model [11], or in a conjectured quark confining phase of non-Abelian gauge theory, we expect restoration of these symmetries.

This paper is concerned with explicitly carrying out canonical quantization of conventional electrodynamics in the temporal gauge. We implement Gauss's law through a limiting procedure maintaining the normalizability of physical states. We do this first for free photons and then find a complete set of physical states for the interacting theory. We set up the perturbation theory on these states and demonstrate equivalence with the textbook treatment of the Coulomb gauge [12]. Throughout, we ignore as mere technical difficulties those problems common to all gauges, such as renormalization of ultraviolet divergences, infrared divergences associated with emission of real photons, convergence of the perturbation series [13], and the existence of the interaction representation [14].

2. Free Photons

In Ref. [9] we briefly discussed free photons in the temporal gauge. There we ignored formal difficulties in normalizing states in the physical subspace of the canonical Hilbert space. Willemsen [10] has treated this problem by smearing Gauss's law and taking a limit. We go through a similar argument here, rephrasing it in terms of the freedom to choose creation and annihilation operators for unphysical longitudinal photons. Taking a limit in this choice, we obtain physical matrix elements from states containing no longitudinal photons. Only matrix elements of gauge invariant operators are well defined in this limit.

We begin with the Lagrangian density

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}, \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{2.2}$$

In the temporal gauge A_0 vanishes and the dynamical coordinates are A_i, $i = 1, 2, 3$. The conjugate momenta are the components of the electric field

$$E_i = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_i)} = F_{0i} = \partial_0 A_i . \tag{2.3}$$

The magnetic field is defined by

$$F_{ij} = \epsilon_{ijk} B_k ,\qquad \mathbf{B} = -\nabla \times \mathbf{A}. \tag{2.4}$$

The Hamiltonian density is

$$\mathscr{H} = E_i\, \partial_0 A_i - \mathscr{L} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2). \tag{2.5}$$

The canonical equal-time commutation relations are

$$[E_i(\mathbf{x}, t), A_j(\mathbf{y}, t)] = -i\delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}). \tag{2.6}$$

Commuting various operators with the Hamiltonian $H = \int d^3x\, \mathscr{H}$ we obtain the equations of motion

$$\partial_0 A_i = i[H, A_i] = E_i , \tag{2.7}$$

$$\partial_0 E_i = i[H, E_i] = \partial_j F_{ji} = (\nabla \times \mathbf{B})_i . \tag{2.8}$$

Note that this procedure does not give Gauss's law $\nabla \cdot \mathbf{E} = 0$ because it does not involve time derivatives. Rather, Eq. (2.8) implies

$$\partial_0(\nabla \cdot \mathbf{E}) = 0 = i[H, \nabla \cdot \mathbf{E}]. \tag{2.9}$$

Because H and $\nabla \cdot \mathbf{E}$ commute, they can formally be simultaneously diagonalized. In Ref. [9] we defined a state $| \Psi\rangle$ to be physical if it satisfied

$$\nabla \cdot \mathbf{E}\, | \Psi\rangle = 0. \tag{2.10}$$

Unfortunately in a Hilbert space realizing Eq. (2.6) the operator $\nabla \cdot \mathbf{E}$ will have a continuous spectrum and consequently its eigenstates are not normalizable. More precisely, note that Eq. (2.6) implies

$$\langle \Psi | [\nabla \cdot \mathbf{E}(\mathbf{x}, t), A_i(y, t)] | \Psi\rangle = -i \frac{\partial}{\partial x_i} \delta^3(\mathbf{x} - \mathbf{y})\langle \Psi | \Psi\rangle. \tag{2.11}$$

If we now demand Eq. (2.10), the norm of $| \Psi\rangle$ must vanish. We will solve this problem by taking a limit on states where $\nabla \cdot \mathbf{E}$ is smeared slightly about zero.

To implement this program we go to momentum space at a fixed time $t = 0$

$$A_i(\mathbf{x}, 0) = \int \frac{d^3k}{(2\pi)^3} \tilde{A}_i(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{x}}, \tag{2.12}$$

$$E_i(\mathbf{x}, 0) = \int \frac{d^3k}{(2\pi)^3} \tilde{E}_i(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{x}}. \tag{2.13}$$

We introduce creation and destruction operators

$$\tilde{A}_i(\mathbf{k}) = \left(\frac{1}{2k_0}\right)^{1/2} (\epsilon_{ij}(\mathbf{k})\, a_j(\mathbf{k}) + (\epsilon_{ij}(-\mathbf{k})\, a_j(-\mathbf{k}))^\dagger), \tag{2.14}$$

$$\tilde{E}_i(\mathbf{k}) = -i\left(\frac{k_0}{2}\right)^{1/2} ((\epsilon^{-1}(\mathbf{k}))_{ij}\, a_j(\mathbf{k}) - ((\epsilon^{-1}(-\mathbf{k}))_{ij}\, a_j(-\mathbf{k}))^\dagger). \tag{2.15}$$

Here $k_0 = |\mathbf{k}|$ and $\epsilon_{ij}(\mathbf{k})$ is a polarization matrix which we will specify further in a moment. The canonical commutation relations are equivalent to

$$[a_i(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = \delta_{ij}\delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.16}$$

Of course, only transverse photons are physical; consequently, we will use the freedom in defining the polarization tensor ϵ_{ij} to eliminate the unwanted longitudinal degrees of freedom. Define the transverse projection operator $P_{ij}(k)$ by

$$P_{ij}(k) = \delta_{ij} - k_i k_j/\mathbf{k}^2. \tag{2.17}$$

We choose the polarization tensor

$$\epsilon_{ij}(k) = P_{ij} + (1/\alpha)(1 - P)_{ij}, \tag{2.18}$$

where α is a parameter that we will soon vary. The transverse part of ϵ is chosen to give the Hamiltonian a simple form in terms of transverse creation and destruction operators. Being unphysical, the longitudinal part of ϵ is essentially arbitrary; indeed, we could have chosen α to be any desired function of $\mathbf{k}$ but for simplicity we take a constant value. The reason for introducing this parameter is that matrix elements involving $\nabla \cdot \mathbf{E}$ are proportional to α

$$\nabla \cdot \mathbf{E}(\mathbf{x}, 0) = \alpha \int \frac{d^3k}{(2\pi)^3} \left(\frac{k_0}{2}\right)^{1/2} (\mathbf{k} \cdot \mathbf{a}(\mathbf{k}) - \mathbf{k} \cdot \mathbf{a}^\dagger(-\mathbf{k}))\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.19}$$

Thus to impose Gauss's law we are interested in the singular limit of vanishing α.

It is a matter of convention whether the α dependence lies in the states or the operators A and E. Putting the α dependence in the states, we consider as our Hilbert space the Fock space built by application of smeared polynomials in $a_i^\dagger(k)$ on the "vacuum" state $|0, \alpha\rangle$ satisfying

$$a_i(\mathbf{k})\,|0, \alpha\rangle = 0, \tag{2.20}$$

$$\langle 0, \alpha\,|\,0, \alpha\rangle = 1 \tag{2.21}$$

for all i and $\mathbf{k}$. Here $a_i(\mathbf{k})$ carries an implicit α dependence from its definition. We now define physical matrix elements to be the $\alpha \to 0$ limit of corresponding matrix elements between states $|\Psi, \alpha\rangle$ containing no longitudinal photons; that is, they satisfy

$$\mathbf{k} \cdot a(\mathbf{k})\,|\Psi, \alpha\rangle = 0. \tag{2.22}$$

In Appendix A we construct a unitary operator that shifts α. Physical quantities follow from a singular limit of this operator on states with no longitudinal photons.

The limit of vanishing α does not exist for general gauge variant matrix elements. Observe, for example, the $1/\alpha^2$ singularity in

$$\begin{aligned}&\langle \alpha, 0 \mid A_i(\mathbf{x}, 0)\, A_j(\mathbf{y}, 0) \mid 0, \alpha\rangle\\ &\quad= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\, \frac{1}{2k_0}\left(\delta_{ij} + \left(\frac{1}{\alpha^2} - 1\right) \frac{k_i k_j}{k^2}\right). \qquad (2.23)\end{aligned}$$

Indeed, it is the $1/\alpha$ in the expression for $A(\mathbf{x}, 0)$ in terms of creation and destruction operators that circumvents the argument below Eq. (2.11) on the nonnormalizability of physical states. Thus, in any calculation the limit $\alpha \to 0$ cannot be taken until one completes all steps involving gauge variant operators and has an expression entirely in terms of physical quantities.

The Hamiltonian expressed in terms of creation and annihilation operators reads

$$\begin{aligned}H = \int \frac{d^3k}{(2\pi)^3}\, k_0 \{P_{ij} a_i^\dagger(k)\, a_j(k)\\ + \tfrac{1}{4}\alpha^2 (1 - P)_{ij} (2a_i^\dagger(k)\, a_j(k) - a_i(k)\, a_j(-k) - a_i^\dagger(k)\, a_j^\dagger(-k))\}, \quad (2.24)\end{aligned}$$

where normal ordering has removed an infinite constant zero-point energy. In the limit of vanishing α the term involving longitudinal photons drops out and we obtain the conventional Hamiltonian for free photons.

The temporal gauge still admits time-independent gauge transformations

$$\mathbf{A}(\mathbf{x}, t) \to \mathbf{A}(\mathbf{x}, t) + \nabla \Lambda(\mathbf{x}), \qquad (2.25)$$

where $\Lambda(\mathbf{x})$ is an arbitrary function of the space coordinates. A unitary operator implementing this transformation is

$$\begin{aligned}U &= \exp\left\{i \int d^3x\, \mathbf{E}(\mathbf{x}, t) \cdot \nabla \Lambda(\mathbf{x})\right\},\\ U\mathbf{A}U^{-1} &= \mathbf{A} + \nabla \Lambda.\end{aligned} \qquad (2.26)$$

Because the Hamiltonian is gauge invariant, U is time independent and is a symmetry operator of the theory. In this paper (in contrast to Ref. [9]) we restrict our discussion to $\Lambda(\mathbf{x})$ vanishing sufficiently rapidly at spatial infinity so that Eq. (2.27) can be partially integrated to give

$$U = \exp\left\{-i \int d^3x\, \Lambda \nabla \cdot \mathbf{E}\right\}. \qquad (2.27)$$

By virtue of Gauss's law $\nabla \cdot \mathbf{E}$ vanishes on physical states. More precisely, we have

$$\langle \Psi', \alpha \mid U \mid \Psi, \alpha\rangle = \langle \Psi', \alpha \mid \Psi, \alpha\rangle + \mathcal{O}(\alpha^2). \qquad (2.28)$$

Thus physical states are symmetric under local gauge transformations. Since $\nabla \cdot \mathbf{E}$ generates such transformations, gauge invariant operators respect Gauss's law and take physical states into physical states.

3. Physical States in the Interacting Theory

We now add charged fermions to the theory and study the Lagrangian density

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi, \tag{3.1}$$

where $\not{D} = \gamma_\mu D_\mu$, D_μ is the covariant derivative

$$D_\mu \Psi = (\partial_\mu - ieA_\mu)\Psi, \tag{3.2}$$

Ψ is a four-component Dirac field, $\bar{\Psi} = \Psi^\dagger \gamma_0$, and the γ matrices satisfy the conventional anticommutation rules

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu}. \tag{3.3}$$

The charge e and mass m are, in principle, divergent due to renormalization effects, but that will not concern us here. In the temporal gauge the canonical coordinates A_i and Ψ with their conjugate momenta E_i and $\Psi^\dagger$ should satisfy the equal-time commutation relations

$$[E_i(\mathbf{x}, t), A_j(\mathbf{y}, t)] = -i\delta_{ij}\delta^3(\mathbf{x} - \mathbf{y}), \tag{3.4}$$

$$[\Psi_\alpha^\dagger(\mathbf{x}, t), \Psi_\beta(\mathbf{y}, t)]_+ = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.5}$$

with all other combinations commuting or anticommuting, as appropriate. The Hamiltonian of the theory is

$$H = \int d^3x \{\tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \bar{\Psi}(-i\boldsymbol{\gamma} \cdot \nabla + m)\Psi + e\mathbf{A} \cdot \mathbf{j}\}, \tag{3.6}$$

where

$$j_\mu = \bar{\Psi}\gamma_\mu\Psi. \tag{3.7}$$

Both H and j_μ involve products of fields at the same point and consequently need more careful definition. As ultraviolet problems are not our prime concern, we relegate to Appendix B a discussion of how a point separation method of definition affects the remaining arguments of this paper.

Equations of motion follow from commutators with the Hamiltonian

$$\partial_0 \mathbf{A} = i[H, \mathbf{A}] = \mathbf{E}, \tag{3.8}$$

$$\partial_0 \mathbf{E} = \nabla \times \mathbf{B} - e\mathbf{j}, \tag{3.9}$$

$$\partial_0 \Psi = i\gamma_0(-i\boldsymbol{\gamma} \cdot \nabla + m)\Psi + ie\gamma_0 \mathbf{A} \cdot \boldsymbol{\gamma}\Psi. \tag{3.10}$$

Current conservation is a consequence of Eq. (3.10)

$$\partial_\mu j_\mu = 0. \tag{3.11}$$

As in the last section, Gauss's law $\nabla \cdot \mathbf{E} - ej_0 = 0$ does not follow as an equation of motion. Rather, from Eqs. (3.9) and (3.11) we conclude

$$\partial_0(\nabla \cdot \mathbf{E} - ej_0) = 0. \tag{3.12}$$

We will again use a limiting procedure to impose Gauss's law on physical states.

For simplicity we will from now on suppress time dependence and work at a fixed time $t = 0$. The Hamiltonian should then be regarded as an operator in a Hilbert space manifesting the canonical commutation relations. Solving the theory consists of finding the eigenvalues and eigenvectors of this operator. Given such a solution, reinsertion of time dependence would be straightforward.

We begin by noting that the interacting Gauss's law differs from the free version where $e = 0$ by an operator displacement on $\nabla \cdot \mathbf{E}$. This shift can be accomplished using the conjugate variable to $\mathbf{E}$. We are thus led to consider the unitary operator

$$V = \exp\left\{\frac{ie}{4\pi}\int d^3x\, d^3y\, \mathbf{A}(\mathbf{x}) \cdot \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} j_0(\mathbf{y})\right\}. \tag{3.13}$$

This operator removes from $\mathbf{E}$ the Coulomb field generated by j_0

$$V\mathbf{E}(\mathbf{x})\, V^{-1} = \mathbf{E}(\mathbf{x}) - \frac{e}{4\pi}\int d^3y\, \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} j_0(\mathbf{y}). \tag{3.14}$$

Using the relation

$$\frac{1}{4\pi}\frac{\partial}{\partial x_i}\frac{(x-y)_i}{|\mathbf{x}-\mathbf{y}|^3} = \delta^3(\mathbf{x}-\mathbf{y}), \tag{3.15}$$

we obtain

$$V\nabla \cdot \mathbf{E}V^{-1} = \nabla \cdot \mathbf{E} - ej_0 . \tag{3.16}$$

The reason for these machinations now appears. If we have a set of states on which $\nabla \cdot \mathbf{E}$ vanishes, then by operating with V on them we obtain a corresponding set of states on which $\nabla \cdot \mathbf{E} - ej_0$ is null. But the whole purpose of the last section was the construction of states with vanishing $\nabla \cdot \mathbf{E}$. Consequently, for a physical matrix, element in the interacting theory we take the $\alpha \to 0$ limit of the corresponding matrix element between states of form $V \mid \Psi, \alpha\rangle$ where α is the longitudinal photon parameter of the last section and $\mid \Psi, \alpha\rangle$ contains arbitrary numbers of transverse photons and fermions, but no longitudinal photons. The expression of the Dirac free field in terms of creation and destruction operators for fermions is standard and need not be repeated here [12].

The interacting theory also possesses an invariance under local gauge transformation of the form

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda, \tag{3.17}$$

but now Ψ must change as well

$$\Psi \to \Psi e^{-ie\Lambda}. \tag{3.18}$$

Using the commutation relation

$$[j_0(\mathbf{x}), \Psi(\mathbf{y})] = -\Psi(\mathbf{x})\, \delta^3(\mathbf{x} - \mathbf{y}), \tag{3.19}$$

one can easily verify that the unitary operator

$$U = \exp \left\{ i \int d^3x (\mathbf{E} \cdot \nabla + ej_0) \Lambda \right\} \tag{3.20}$$

implements this transformation in the sense that

$$U\mathbf{A}U^{-1} = \mathbf{A} + \nabla \Lambda, \tag{3.21}$$

$$U\Psi U^{-1} = \Psi e^{-ie\Lambda}. \tag{3.22}$$

Again assuming that $\Lambda(\mathbf{x})$ vanishes at spatial infinity so that Eq. (3.20) can be partially integrated, we see that Gauss's law generates gauge transformations for the interacting theory as well

$$U = \exp \left\{ -i \int d^3x\, \Lambda (\nabla \cdot \mathbf{E} - ej_0) \right\}. \tag{3.23}$$

Thus all the discussion of the previous section on invariance of physical states under time-independent gauge transformations repeats itself.

The operator V of Eq. (3.13) serves to create a Coulomb field around fermions. This can be seen by using Eq. (3.19) to show

$$\chi(\mathbf{x}) \equiv V\Psi(\mathbf{x})\, V^{-1} = \Psi(x) \exp \left\{ \frac{ie}{4\pi} \int d^3y \frac{(\mathbf{x} - \mathbf{y})}{| x - y |^3} \cdot \mathbf{A}(\mathbf{y}) \right\}. \tag{3.24}$$

The exponential factor is an operator that displaces $\mathbf{E}$ by just the Coulomb field of a point charge at $\mathbf{x}$. Note also that this combination is gauge invariant; indeed, a local gauge transformation in Eq. (3.24) shifts A by a derivative term which upon partial integration and use of Eq. (3.15) exactly cancels the phase given $\Psi(x)$. This means that smeared polynomials in $\chi(\mathbf{x})$ do not take states out of the physical Hilbert space and, therefore, represent physical fermion operators.

We have defined the operator V in such a way that it only involves the longitudinal part of $\mathbf{A}$. This follows from rewriting V in the form

$$V = \exp \left\{ \frac{ie}{4\pi} \int d^3x\, d^3y\, j_0(\mathbf{x}) \frac{1}{| \mathbf{x} - \mathbf{y} |} \nabla \cdot \mathbf{A}(\mathbf{y}) \right\}. \tag{3.25}$$

We could have replaced the kernel $(\mathbf{x} - \mathbf{y})/| x - y |^3$ in Eq. (3.13) with any function of $\mathbf{x} - \mathbf{y}$ satisfying Eq. (3.15). However, use of a purely longitudinal form simplifies perturbation theory by preserving the transverse part of the Hamiltonian.

4. Perturbation Theory

Having obtained a prescription for physical matrix elements, we now wish to set up a perturbation theory for diagonalization of the Hamiltonian. We start with the definition

$$H_0 = \int d^3x\{\tfrac{1}{2}(\mathbf{E}_T{}^2 + \mathbf{B}^2) + \bar{\Psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\Psi\}. \tag{4.1}$$

Here $\mathbf{E}_T$ is the transverse part of $\mathbf{E}$, i.e., only the creation and destruction operators for transverse photons are included. This operator is diagonalized by the free particle states created by the operators of Section 2 plus standard free Dirac creation operators. However, as it stands, H_0 is not diagonalized by the states of the last section used to implement Gauss's law for the interacting theory. Also, when the coupling e does not vanish, H_0 is not a gauge invariant operator and thus it takes states out of the physical subspace. An operator that avoids these difficulties is VH_0V^{-1}. This combination is gauge invariant by virtue of the discussion at the end of the last section, and it takes the usual free particle form on the states of the last section. We thus are led to define an interaction Hamiltonian H_I with the relation

$$H = VH_0V^{-1} + H_I. \tag{4.2}$$

Since H and VH_0V^{-1} are both gauge invariant, H_I is as well.

We will now express H_I in terms of gauge invariant fields and show that it is equal to the interaction Hamiltonian of the usual Coulomb gauge treatment of quantum electrodynamics [12]. Note that matrix elements of products of the gauge invariant χ fields between the physical states of the last section equal the corresponding matrix elements of $\Psi(x)$ between free fermion states. This ensures that in the perturbative expansion one can use free propagators for the fermions. Combining Eqs. (3.6), (4.1), and (4.2) we obtain

$$H_I = \int d^3x\{\tfrac{1}{2}\mathbf{E}_L{}^2 + \bar{\Psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\Psi - \bar{\chi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\chi + e\mathbf{j}\cdot\mathbf{A}\}, \tag{4.3}$$

where $\mathbf{E}_L = \mathbf{E} - \mathbf{E}_T$ is the longitudinal part of the electric field. From Eq. (3.24) we obtain

$$\bar{\Psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\Psi - \bar{\chi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\chi = -e\mathbf{j}\cdot\mathbf{A}_L, \tag{4.4}$$

where we have used the position space representation of the longitudinal part of A

$$\mathbf{A}_L = \frac{1}{4\pi}\boldsymbol{\nabla}\int d^3y\,\frac{(\mathbf{x}-\mathbf{y})}{|x-y|^3}\cdot\mathbf{A}(y). \tag{4.5}$$

In obtaining Eq. (4.4) we have naively taken

$$\mathbf{j} = V\mathbf{j}V^{-1}. \tag{4.6}$$

As $\mathbf{j}$ is electrically neutral and the purpose of V is to create Coulomb fields about charged operators, Eq. (4.6) physically makes sense. However, caution is in order because of possible Schwinger terms [15] arising when $\mathbf{j}$ is carefully defined. The discussion of point separation in Appendix B justifies Eq. (4.4) with a gauge invariant definition of $\mathbf{j}$.

Using Eq. (4.4) in (4.3) we obtain

$$H_I = \int d^3x \{\tfrac{1}{2}\mathbf{E}_L{}^2 + e\mathbf{A}_T \cdot \mathbf{j}\}. \tag{4.7}$$

With Gauss's law we can rewrite $\mathbf{E}_L$ in terms of j_0

$$\begin{aligned} \mathbf{E}_L(\mathbf{x}) &= -\frac{1}{4\pi} \nabla \int d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|} \nabla \cdot \mathbf{E}(\mathbf{y}) \\ &= -\frac{e}{4\pi} \nabla \int d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|} j_0(\mathbf{y}) + \mathcal{O}(\alpha), \end{aligned} \tag{4.8}$$

where the $\mathcal{O}(\alpha)$ term is of order α when considered in matrix elements between the α states of the last section. Inserting (4.8) in (4.7) yields

$$H_I = \int d^3x\, e\mathbf{A}_T \cdot \mathbf{j} + \int d^3x\, d^3y \frac{e^2}{4\pi} j_0(\mathbf{x}) \frac{1}{|\mathbf{x}-\mathbf{y}|} j_0(\mathbf{y}) + \mathcal{O}(\alpha). \tag{4.9}$$

The limit $\alpha \to 0$ is now immediate for gauge invariant Green's functions. The theory has been expressed entirely in terms of physical degrees of freedom; consequently longitudinal photons can be forgotten and perturbation theory proceeds in a standard way. Note that H_I in Eq. (4.9) is exactly the interaction Hamiltonian that arises in the conventional Coulomb gauge treatment [12]. Thus the two approaches are equivalent.

5. Concluding Remarks

We have demonstrated the equivalence of the temporal and Coulomb gauges for a perturbative treatment of canonically quantized electrodynamics. Given the nonperturbative problems with other gauges, it would be desirable to have a complete canonical treatment of non-Abelian gauge theory in the temporal gauge. Imposition of Gauss's law in this case is a considerably more difficult task. The desired constraints involve a noncommuting set of nonlinear combinations of the gauge fields. One approach to finding states satisfying these constraints is that of Goldstone and Jackiw [8]; however, their technique does not lend itself to an expansion in the coupling constant. A perturbative treatment should be important in the study of an unconfined electrodynamicslike phase of free gauge mesons. Indeed, understanding such a phase ought to precede attempts to avoid it.

APPENDIX A: Shifting the Parameter α

Consider the formally unitary operater

$$T(\lambda) = \exp\left\{\frac{-i\lambda}{4\pi}\int d^3x\, d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{1}{2}[\nabla\cdot\mathbf{E}(\mathbf{x}), \nabla\cdot\mathbf{A}(\mathbf{y})]_+\right\}, \tag{A1}$$

where $[A, B]_+$ denotes the anticommutation $AB + BA$. Being defined in terms of longitudinal fields, T commutes with transverse fields. From the canonical commutation relations we obtain

$$\frac{d}{d\lambda} T^{-1}\nabla\cdot\mathbf{E}T = -T^{-1}\nabla\cdot\mathbf{E}T. \tag{A2}$$

This is easily solved

$$T^{-1}\nabla\cdot\mathbf{E}T = e^{-\lambda}\nabla\cdot\mathbf{E}. \tag{A3}$$

Similarly we obtain

$$T^{-1}\nabla\cdot\mathbf{A}T = e^{+\lambda}\nabla\cdot\mathbf{A}. \tag{A4}$$

Thus we conclude that the operator T multiplies α by $e^{-\lambda}$

$$T\,|\,\Psi, \alpha\rangle = |\,\Psi, e^{-\lambda}\alpha\rangle. \tag{A5}$$

The limit of vanishing α corresponds to infinite λ.

APPENDIX B: Point Separation

The Hamiltonian of Eq. (3.6) involves products of fields at the same space-time point. In general such products are singular and need more precise definition. Since the interaction is linear in the photon field, any simple definition of the kinetic terms $\frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2$ will do; a normal ordering with respect to the creation and destruction operators of Section 2 is probably the simplest method. The Fermi fields are potentially more troublesome due to the well-known Schwinger terms [15] appearing in commutators among components of the electromagnetic current.

In this Appendix we treat the fermion operators with a point separation technique. We do this in a manner that preserves the gauge invariance of the respective operators. Upon separating the coordinates of Ψ and $\bar{\Psi}$ in an operator such as j_μ, we must multiply by an operator that creates the dipole field that would physically accompany any such real separation. Letting $\boldsymbol{\epsilon}$ be an infinitesimal vector that goes to zero at the end of any calculation, we modify definition (3.7) to read

$$j_\mu = \exp\left\{\frac{-ie}{4\pi}\int d^3y \left(\frac{1}{|\mathbf{x}+\boldsymbol{\epsilon}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\nabla\cdot\mathbf{A}(y)\right\}\bar{\Psi}(\mathbf{x}+\boldsymbol{\epsilon})\,\gamma_\mu\Psi(\mathbf{x}), \tag{B1}$$

where time dependence is suppressed by working at fixed $t = 0$. The operator V used to define physical states is used only through its commutation relations with other operators. Consequently, the ϵ used in defining V can be taken to zero immediately. Adopting this point of view, we can let the operator V create the dipole field in Eq. (B1)

$$\begin{aligned} j_\mu(\mathbf{x}) &= V\bar{\Psi}(\mathbf{x} + \epsilon)\, \gamma_\mu \Psi(\mathbf{x})\, V^{-1} \\ &= \bar{\chi}(\mathbf{x} + \epsilon)\, \gamma_\mu \chi(\mathbf{x}). \end{aligned} \tag{B2}$$

The kinetic terms in the Hamiltonian also require definition. Here the derivative should not act on the dipole field because the gauge variant term associated with this derivative combines with $\mathbf{j} \cdot \mathbf{A}$ in a gauge invariant way. Thus, in H we define

$$\begin{aligned} \bar{\Psi}(-i\boldsymbol{\gamma} \cdot \nabla + m)\Psi \to \exp &\left\{\frac{-ie}{4\pi} \int d^3y \left(\frac{1}{|\mathbf{x} + \epsilon - \mathbf{y}|} - \frac{1}{|\mathbf{x} - \mathbf{y}|}\right) \nabla \cdot \mathbf{A}(y)\right\} \\ &\times \bar{\Psi}(\mathbf{x} + \epsilon)(-i\boldsymbol{\gamma} \cdot \nabla + m)\, \Psi(\mathbf{x}). \end{aligned} \tag{B3}$$

This is a valid definition of the fermion kinetic terms in the full Hamiltonian, but when we want a diagonal VH_0V^{-1} we should use for H_0 the free fermion kinetic energy where e is taken to zero. Thus, we define

$$H_0 = \int d^3x \{\tfrac{1}{2} :(\mathbf{E}_\perp{}^2 + \mathbf{B}^2): + \bar{\Psi}(\mathbf{x} + \epsilon)(-i\boldsymbol{\gamma} \cdot \nabla + m)\, \Psi(\mathbf{x})\} \tag{B4}$$

with no dipole field for the bare fermions. Here the symbols : : denote normal ordering with respect to the free photon operators of Section 2. The interaction Hamiltonian then becomes

$$\begin{aligned} H_I = \int d^3x &\left[\frac{1}{2} :\mathbf{E}_L{}^2: + \exp\left\{\frac{-ie}{4\pi} \int d^3y \left(\frac{1}{|\mathbf{x} + \epsilon - \mathbf{y}|} - \frac{1}{|\mathbf{x} - \mathbf{y}|}\right) \nabla \cdot \mathbf{A}\right\}\right. \\ &\left.\times\, \bar{\Psi}(\mathbf{x} + \epsilon)(-i\boldsymbol{\gamma} \cdot \nabla + m)\, \Psi(\mathbf{x}) - \bar{\chi}(\mathbf{x} + \epsilon)(-i\boldsymbol{\gamma} \cdot \nabla + m)\, \chi(\mathbf{x}) + e\mathbf{j} \cdot \mathbf{A}\right]. \end{aligned} \tag{B5}$$

The manipulations giving Eq. (4.4) now follow with the dipole factors giving exactly the gauge invariant $\mathbf{j}$ of Eq. (B1). Note that Eq. (B2) implies that in the perturbation series we should use the χ propagator between physical states, but as discussed below Eq. (4.3) this equals the bare Ψ propagator between states of the free theory.

We finally note that both $\mathbf{j}$ and H_0 as defined here have vacuum expectation values. Appropriate counterterms can remove these; then the resulting definitions in the $\epsilon \to 0$ limit correspond to conventional normal ordering.

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