PHYSICAL REVIEW D VOLUME 16, NUMBER 10 15 NOVEMBER 1977

Topological tunneling and Goldstone gluons*

Michael Creutz and Thomas N. Tudron
Brookhaven National Laboratory, Upton, New York 11973
(Received 7 July 1977)

Canonically quantizing in the temporal gauge $A_0 = 0$, we study the symmetry properties of the gauge theory vacuum under time-independent gauge transformations. A quantum-electrodynamics-like unconfined phase exhibits spontaneously broken symmetry under gauge transformations that do not vanish at spatial infinity. In a confining phase this symmetry should be restored. When in the unconfined phase, assuming it exists, of a theory possessing topologically nontrivial gauge transformations, the physical Hilbert space will admit a discrete symmetry operation related to a tunneling process between discrete classical vacuums. In the confined phase, this symmetry becomes part of a continuous gauge symmetry. We discuss in detail the solvable theories of free photons and the two-dimensional Schwinger model. We also give some nonrigorous arguments that the phase θ associated with the tunneling process may have no physical significance in four-dimensional space-time.

I. INTRODUCTION

A theory of quarks interacting with non-Abelian gauge fields has recently become a popular candidate to describe strong-interaction dynamics.[1] This provides the pleasing possibility that all particle interactions from gravity to the nuclear force are gauge theories. As usually discussed, the strong-interaction theory differs in one important respect from other applications of field theory in that no free particles correspond directly to the fundamental fields appearing in the Lagrangian. Indeed, the quarks and vector "gluons" are hoped to be "confined" in the physical hadrons which are gauge-singlet bound states.

Little evidence exists that the non-Abelian gauge theory does indeed confine. Renormalization-group arguments yield a large effective coupling for large quark separation.[1] This suggests that perturbation theory may break down at large distances, which would explain the lack of perturbative evidence for confinement.[2] Eliminating ultraviolet divergences with the artifice of a lattice, Wilson has investigated the strong-coupling limit of gauge theories.[3,4] In this limit the theory becomes one of quarks connected by strings, and confinement is natural. Using mean-field-theory arguments, Balian, Drouffe, and Itzykson have suggested that in enough space-time dimensions gauge theories undergo a phase transition as the coupling constant is varied.[5] This transition is between the confined strong-coupling phase and a quantum-electrodynamics-like unconfined phase. Migdal has given approximate arguments that four-dimensional space-time represents a critical case where Abelian theories undergo this transition while non-Abelian ones remain confined.[6]

Recently Polyakov suggested that certain classical solutions to non-Abelian gauge theory in Euclidean space-time may have some bearing on confinement.[7] Qualitatively, such configurations may result in a disordering forcing the theory into the strong-coupling phase. These "pseudoparticle" solutions are indicative of a tunneling process between distinct classical ground states of the theory in Minkowski space and formulated in the temporal gauge $A_0 = 0$.[8] In this paper we use canonical techniques to study further the consequences of this tunneling for the confined and unconfined phases.

The unconfined phase of a gauge theory is signaled by the noninvariance of the vacuum state under gauge transformations that do not vanish at spatial infinity. This concept has been extensively discussed previously in the Lorentz gauge $\partial_\mu A_\mu = 0$.[9] We reformulate this idea in the temporal gauge. The massless vector mesons are Goldstone bosons in this phase of spontaneous symmetry breaking. In non-Abelian gauge theories there are gauge transformations that change the topology of the classical vacuum. These generate a discrete symmetry in the physical Hilbert space of the unconfined phase. Unfortunately, we find no obvious contradiction in having such a symmetry in the unconfined phase; thus, we do not have an argument for confinement.

In the confined phase of a theory we argue that symmetry under all gauge transformations will be restored. Then the topologically nontrivial gauge transformations become part of a continuous symmetry, and the analogy to a tunneling phenomenon becomes obscure.

In Sec. II we canonically quantize the solvable theory of free photons in the temporal gauge. We construct operators that generate time-independent gauge transformations and study their effect on the vacuum. In Sec. III we investigate SU(2) non-Abelian gauge theory where topologically nontrivial gauge transformations exist. Section IV discusses the perturbation-theory vacuum and corrections necessary to take account of the tunneling process. In Sec. V we use the Schwinger model of two-dimensional quantum electrodynamics as an example of a confined theory with nontrivial topological properties. Section VI gives a nonrigorous discussion on the tunneling process and background electric fields. Here we argue that the phase θ encountered in the tunneling process may have no physical effects in four-dimensional space-time. Section VII contains some concluding remarks.

II. FREE PHOTONS IN THE TEMPORAL GAUGE

In this section we study the canonical quantization of free photons in the temporal gauge $A_0=0$. This section is essentially a reworking in this gauge of the work of Brandt and Ng and Ferrari and Picasso.[9] We begin with the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}, \tag{2.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu . \tag{2.2}$$

The dynamical coordinates are A_i, $i=1,2,3$, and their conjugate momenta are just the components of the electric field

$$\pi_i=\frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)}=F_{0i}=\partial_0 A_i\equiv E_i . \tag{2.3}$$

The magnetic field is given by

$$\begin{aligned} F_{ij}&=\partial_i A_j-\partial_j A_i\equiv\epsilon_{ijk}B_k ,\\ \vec{B}&=-\vec{\nabla}\times\vec{A} . \end{aligned} \tag{2.4}$$

The Hamiltonian density is

$$\mathcal{H}=E_i\partial_0 A_i-\mathcal{L}=\tfrac{1}{2}E_i^{\,2}+\tfrac{1}{4}F_{ij}F_{ij}=\tfrac{1}{2}(E^2+B^2) . \tag{2.5}$$

The canonical equal-time commutation relations are

$$[E_i(\vec{x},t),A_j(\vec{y},t)]=-i\delta_{ij}\delta^3(\vec{x}-\vec{y}) . \tag{2.6}$$

By commuting operators with the Hamiltonian we obtain some of the equations of motion ($H=\int d^3x\mathcal{H}$)

$$\partial_0 A_i=i[H,A_i]=E_i , \tag{2.7}$$

$$\partial_0 E_i=i[H,E_i]=\partial_j F_{ji}=(\vec{\nabla}\times\vec{B})_i . \tag{2.8}$$

Note that from the Hamiltonian we cannot obtain Gauss's law, $\vec{\nabla}\cdot\vec{E}=0$, because it does not involve time derivatives. Rather, Eq. (2.8) only implies

$$\partial_0(\vec{\nabla}\cdot\vec{E})=0=i[H,\vec{\nabla}\cdot\vec{E}] . \tag{2.9}$$

This means that $\vec{\nabla}\cdot\vec{E}$ and H can be simultaneously diagonalized. We say a state $|\psi\rangle$ is physical if it satisfies the restriction

$$\vec{\nabla}\cdot\vec{E}\,|\psi\rangle=0 . \tag{2.10}$$

All operators that respect Gauss's law, i.e., those that commute with $\vec{\nabla}\cdot\vec{E}$, will leave us in the sector of physical states.

To find the spectrum of this theory we go to momentum space,

$$\begin{aligned} A_i(\vec{x},t)&=\int\frac{d^3k}{(2\pi)^3 2k_0}\tilde{A}_i(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} ,\\ E_i(\vec{x},t)&=\int\frac{d^3k}{(2\pi)^3 2k_0}\tilde{E}_i(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} , \end{aligned} \tag{2.11}$$

where $k_0\equiv|\vec{k}|$. Equations (2.7) and (2.8) now become

$$\partial_0\tilde{A}_i=\tilde{E}_i , \tag{2.12}$$

$$\partial_0\tilde{E}_i=-\vec{k}^2P_{ij}\tilde{A}_j , \tag{2.13}$$

where P_{ij} is the transverse projection operator

$$P_{ij}=\delta_{ij}-k_ik_j/k^2 . \tag{2.14}$$

The physical constraint in Eq. (2.10) reads

$$(1-P)_{ij}\tilde{E}_j\,|\psi\rangle=0 . \tag{2.15}$$

Creation and destruction operators for physical photons are defined by

$$\begin{aligned} P_{ij}\tilde{A}_j&=a_i(\vec{k},t)+a_i^\dagger(-\vec{k},t) ,\\ P_{ij}\tilde{E}&=-ik_0[a_i(\vec{k},t)-a_i^\dagger(-\vec{k},t)] . \end{aligned} \tag{2.16}$$

Clearly we have

$$P_{ij}a_j=a_i . \tag{2.17}$$

Equations (2.12) and (2.13) become

$$\partial_0 a_i=-ik_0a_i , \tag{2.18}$$

which implies

$$a_i(\vec{k},t)=e^{-ik_0t}a_i(\vec{k}) . \tag{2.19}$$

The commutation relations in Eq. (2.6) imply

$$[a_i(\vec{k}),a_j^\dagger(\vec{k}')]=P_{ij}2k_0(2\pi)^3\delta^3(\vec{k}-\vec{k}') . \tag{2.20}$$

Straightforward manipulation gives

$$H=\int\frac{d^3k}{(2\pi)^3 2k_0}\left[k_0a_i^\dagger a_i+\frac{1}{4k_0}\tilde{E}_i(\vec{k})(1-P)_{ij}\tilde{E}_j(-\vec{k})\right] , \tag{2.21}$$

where an infinite zero-point energy has been dropped. The physical vacuum $|0\rangle$ is defined uniquely to a phase by

$$a_i\,|0\rangle=0 , \tag{2.22}$$

$$(1-P)_{ij}\tilde{E}_j\,|0\rangle=0 , \tag{2.23}$$

$$\langle 0|0\rangle=1 . \tag{2.24}$$

From Eq. (2.21) it is easily seen that this is the lowest energy state satisfying the constraint Eq. (2.15).

The temporal gauge still leaves the freedom of performing time-independent gauge transformations of the form

$$A_i(\vec{x}) \to A_i(\vec{x}) + \nabla_i \Lambda(\vec{x}), \tag{2.25}$$

where $\Lambda(\vec{x})$ is an arbitrary function of the space coordinates. Using the canonical commutation relations, we can easily construct an operator producing this transformation,

$$UA_iU^{-1} = A_i + \nabla_i\Lambda,$$
$$U = \exp\left(i\int d^3x E_i(\vec{x})\nabla_i\Lambda(\vec{x})\right). \tag{2.26}$$

We define a "local" gauge transformation to be one for which $\Lambda(\vec{x})$ vanishes at spatial infinity rapidly enough that we can neglect surface terms and partially integrate Eq. (2.26) to obtain

$$U = \exp\left(-i\int d^3x\, \Lambda\vec{\nabla}\cdot\vec{E}\right). \tag{2.27}$$

By virtue of Gauss's law, Eq. (2.10), $\vec{\nabla}\cdot\vec{E}$ vanishes on physical states. In particular, we have

$$U|0\rangle = |0\rangle. \tag{2.28}$$

Thus, the vacuum is invariant under local gauge transformations. Since $\vec{\nabla}\cdot\vec{E}$ is effectively the generator of such transformations, those operators which respect Gauss's law are the set of gauge-invariant operators. Any gauge-invariant operator will take a physical state into a physical state.

We now turn to study gauge transformations that do not vanish at infinity. In particular, consider

$$\Lambda(\vec{x}) = \lambda x_\alpha \tag{2.29}$$

such that

$$A_i \to A_i + \lambda\delta_{i\alpha}. \tag{2.30}$$

The important feature here is that we change A_i by a finite amount over an infinite volume. Equation (2.26) still gives the operator that produces this transformation

$$U = \exp\left(i\lambda\int d^3x E_\alpha\right). \tag{2.31}$$

We wish to study the action of this operator on the vacuum state. It is convenient to introduce at this point an infrared cutoff ϵ and study

$$U(\epsilon) = \exp\left(i\lambda\int d^3x E_\alpha e^{-\epsilon^2\vec{x}^2}\right). \tag{2.32}$$

In Appendix A we show

$$\langle 0|U(\epsilon)|0\rangle = \exp\left(-\frac{\lambda^2\pi}{6\epsilon^2}\right) \underset{\epsilon\to 0}{\longrightarrow} 0. \tag{2.33}$$

As $\epsilon\to 0$, U becomes a symmetry operation that commutes with the Hamiltonian. However, Eq. (2.33) shows that the vacuum does not respect the symmetry. Indeed, if we generate our Hilbert space by applying to the vacuum gauge-invariant field combinations smeared with test functions of compact support, then U takes the vacuum out of the Hilbert space. Thus we have an example of spontaneous symmetry breaking. As it is a continuous symmetry, we expect massless Goldstone particles, a role played by the physical photon. More precisely, for ϵ small but nonvanishing, $U(\epsilon)|0\rangle \equiv |\epsilon\rangle$ is a state in the physical Hilbert space because the physical constraint commutes with $U(\epsilon)$ and $e^{-\epsilon^2x^2}$ is a suitable test function. This state has an energy expectation arbitrarily close to the vacuum energy if ϵ is made small enough, yet its overlap with the vacuum is also arbitrarily small. The presence of this state shows that there cannot be a gap in the spectrum of the theory and therefore there is a massless particle, the photon.

Although α in Eq. (2.29) can take on three values, only the two transverse photons represent Goldstone bosons. To see this, apply the cutoff keeping the gauge transformation purely longitudinal

$$U_L = \exp\left\{i\int d^3x E_i\nabla_i[\Lambda(\vec{x})e^{-\epsilon^2\vec{x}^2}]\right\}. \tag{2.34}$$

For finite ϵ we can partially integrate and apply the constraint condition to obtain

$$U_L|0\rangle = |0\rangle, \tag{2.35}$$

so we do not obtain longitudinal massless particles.

Note that the effect of the gauge transformation with $\Lambda(\vec{x})$ given as in (2.29) is to change the fields by a nonzero amount over an infinite volume of space. In fact, if Λ goes as some positive power of $|\vec{x}|$ for large $|\vec{x}|$, then this same result will hold—the transformation takes the vacuum out of the Hilbert space. But if Λ is chosen to go to zero as some negative power for large $|\vec{x}|$, the vacuum will respect the corresponding gauge symmetry. For reasons to become clear in Sec. III, it is convenient to consider $\Lambda(\vec{x})$ which are on the borderline between these two cases. For example, consider

$$\Lambda(\vec{x}) = \frac{\lambda x_3}{(x^2+\rho^2)^{1/2}}, \tag{2.36}$$

with ρ an arbitrary parameter. This is an example where $\Lambda(\vec{x})$ goes to a direction-dependent constant as $|\vec{x}|$ goes to infinity. In Appendix A we show for this Λ

$$\langle 0|U|0\rangle = \exp(-\lambda^2 \times 0.55\ldots)\,. \tag{2.37}$$

Again, this can be interpreted as an indication of the Goldstone phenomenon because the vacuum is not invariant.

We have argued that the massless vector mesons of a gauge theory quantized in the temporal gauge are Goldstone bosons associated with spontaneously broken gauge invariance. If we desire a gauge theory with confinement and no massless vector gauge particles, this symmetry should be restored. We expect that gauge transformations that change the gauge field a finite amount over an infinite volume are symmetries of the confined vacuum but not in the unconfined case. Thus the symmetry properties of the vacuum provide a criterion for confinement.

III. SU(2) GAUGE THEORY

For reasons of simplicity we discuss the gauge group SU(2) rather than the physically relevant SU(3). We also ignore here the quark fields on the assumption that confinement, if it occurs, should appear in the pure gauge field sector of the theory. Thus we study the Lagrangian (repeated indices are summed)

$$\mathcal{L} = -\tfrac{1}{4}F^\alpha_{\mu\nu}F^\alpha_{\mu\nu}\,, \tag{3.1}$$

where

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - e\epsilon^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu\,. \tag{3.2}$$

The index α runs from one to three, $\epsilon^{\alpha\beta\gamma}$ is the totally antisymmetric tensor with $\epsilon^{123}=1$ and e is the coupling constant of the theory. It is often convenient to use a matrix notation with

$$A_\mu = \tfrac{1}{2}A^\alpha_\mu\sigma^\alpha, \quad F_{\mu\nu} = \tfrac{1}{2}F^\alpha_{\mu\nu}\sigma^\alpha\,, \tag{3.3}$$

$$A^\alpha_\mu = \mathrm{Tr}(\sigma^\alpha A_\mu), \quad F^\alpha_{\mu\nu} = \mathrm{Tr}(\sigma^\alpha F_{\mu\nu})\,, \tag{3.4}$$

where σ^α are the 2×2 Pauli matrices satisfying

$$\begin{aligned}[]
[\sigma^\alpha, \sigma^\beta] &= 2i\epsilon^{\alpha\beta\gamma}\sigma^\gamma\,,\\
[\sigma^\alpha, \sigma^\beta]_+ &= 2\delta^{\alpha\beta}\,.
\end{aligned} \tag{3.5}$$

In this matrix notation we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]\,, \tag{3.6}$$

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})\,. \tag{3.7}$$

The gauge invariance of this theory is described by considering an arbitrary mapping $g(\vec{x})$ of space into the group SU(2). Then the Lagrangian density is invariant under

$$A_\mu \to gA_\mu g^{-1} + \frac{i}{e}(\partial_\mu g)g^{-1}\,. \tag{3.8}$$

With this change $F_{\mu\nu}$ transforms as

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1}\,. \tag{3.9}$$

Given $G_{ij}(g)$ some matrix representation of SU(2), we can consider fields that transform under the above gauge transformation as this representation

$$\phi_i(\vec{x}) \to G_{ij}(g(\vec{x}))\phi_j(\vec{x})\,. \tag{3.10}$$

If SU(2) elements are parameterized in the form

$$g = e^{i\vec{\omega}\cdot\vec{\sigma}/2}\,, \tag{3.11}$$

one can write

$$G_{ij}(g) = [\exp(i\vec{\omega}\cdot\vec{v})]_{ij}\,, \tag{3.12}$$

where v^α_{ij} are matrices generating the representation and satisfying

$$[v^\alpha, v^\beta] = i\epsilon^{\alpha\beta\gamma}v^\gamma\,. \tag{3.13}$$

It is well known that one can define a covariant derivative of the field ϕ_i by

$$(D_\mu\phi)_i = \partial_\mu\phi_i + ieA^\alpha_\mu v^\alpha_{ij}\phi_j \tag{3.14}$$

and that this transforms under gauge transformations as

$$(D_\mu\phi)_i \to G_{ij}(D_\mu\phi)_j\,. \tag{3.15}$$

It is easily checked that

$$[D_\mu, D_\nu]\phi = ieF^\alpha_{\mu\nu}v^\alpha\phi. \tag{3.16}$$

For $F^\alpha_{\mu\nu}$ the relevant representation is the adjoint representation given by

$$g^{-1}\sigma^\alpha g = G^{\alpha\beta}\sigma^\beta\,. \tag{3.17}$$

The generators are

$$v^\alpha_{\beta\gamma} = -i\epsilon^{\alpha\beta\gamma}\,. \tag{3.18}$$

Consequently, we have

$$(D_\mu F_{\nu\rho})^\alpha = \partial_\mu F^\alpha_{\nu\rho} - e\epsilon^{\alpha\beta\gamma}A^\beta_\mu F^\gamma_{\nu\rho}\,. \tag{3.19}$$

The classical Euler-Lagrange equations take the simple form

$$(D_\mu F_{\mu\nu})^\alpha = 0\,. \tag{3.20}$$

We now go to the temporal gauge $A^\alpha_0 = 0$ and proceed to quantize the theory. We work at a fixed time and suppress time dependence in what follows. The dynamical variables are A^α_i and their conjugate momenta are

$$\pi^\alpha_i = F^\alpha_{0i} \equiv E^\alpha_i\,. \tag{3.21}$$

The Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}E^\alpha_i E^\alpha_i + \tfrac{1}{4}F^\alpha_{ij}F^\alpha_{ij}\,. \tag{3.22}$$

Using the canonical equal-time commutation relations

$$[E_i^\alpha(\vec{x}), A_j^\beta(\vec{y})] = -i\delta_{ij}\delta^{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{3.23}$$

in commutators with the Hamiltonian, one can obtain the equations of motion

$$\partial_0 A_i^\alpha = E_i^\alpha\,, \tag{3.24}$$

$$\partial_0 E_i^\alpha = (D_j F_{ji})^\alpha\,. \tag{3.25}$$

As in the Abelian case, Gauss's law $(D_i E_i)^\alpha = 0$ does not follow from the Hamiltonian; rather, we only obtain

$$\partial_0 (D_i E_i)^\alpha = 0\,. \tag{3.26}$$

Again this means that $D_i E_i$ can be simultaneously diagonalized with the Hamiltonian; so, we define physical states by

$$(D_i E_i)^\alpha |\psi\rangle = 0\,. \tag{3.27}$$

A vacuum can be defined as the lowest eigenstate of the Hamiltonian under this constraint condition. With e nonzero we have an interacting theory, and hence little is known about this state.

In the temporal gauge we are still left with the possibility of performing time-independent gauge transformations. These are realized by considering a $g(\vec{x})$ in Eq. (3.8) that depends only on space coordinates

$$A_i(\vec{x}) \to g(\vec{x}) A_i(\vec{x}) g^{-1}(\vec{x}) + \frac{i}{e}[\partial_i g(\vec{x})] g^{-1}(\vec{x})\,. \tag{3.28}$$

We wish to construct a unitary operator that causes this transformation

$$U A_i U^{-1} = g A_i g^{-1} + \frac{i}{e}(\partial_i g) g^{-1}\,. \tag{3.29}$$

Parameterize $g(\vec{x})$ as

$$g(\vec{x}) = \exp\left[i\frac{\vec{\sigma}}{2}\cdot\vec{\omega}(\vec{x})\right]. \tag{3.30}$$

We show in Appendix B that with $\vec{\omega}$ defined in such a way, an operator U that implements the desired gauge transformation is

$$U = \exp\left[-\frac{i}{e}\int d^3x\, E_i^\alpha (D_i\omega)^\alpha\right], \tag{3.31}$$

where we have made the formal definition

$$D_i\omega^\alpha = \partial_i\omega^\alpha - e\epsilon^{\alpha\beta\gamma} A_i^\beta \omega^\gamma\,. \tag{3.32}$$

Note the similarity between Eqs. (3.31) and (2.27). Because the Hamiltonian is gauge invariant, we have

$$\partial_0 U = i[H, U] = 0\,. \tag{3.33}$$

In the Abelian case we divided the time-independent gauge transformations into two types, those that left $A(\vec{x})$ unchanged at infinite $|\vec{x}|$ and those that did not. For the SU(2) case it is useful to subdivide the first class further. The number of times an arbitrary mapping $g(\vec{x})$ covers the group SU(2) is given by[10]

$$n = \frac{1}{24\pi^2}\int d^3x\, \epsilon_{ijk} \mathrm{Tr}[(\nabla_i g) g^{-1} (\nabla_j g) g^{-1} (\nabla_k g) g^{-1}]\,. \tag{3.34}$$

This number is an integer for continuous $g(\vec{x})$ that go to the identity as $|\vec{x}|$ goes to infinity. We now consider the following three categories of gauge transformations:

(Ia) This class consists of those $g(\vec{x})$ where $g(x) \to 1$ as $|\vec{x}| \to \infty$ and where the expression in Eq. (3.34) vanishes.

(Ib) Here we include those $g(\vec{x})$ with $g(\vec{x}) \to 1$ as $|\vec{x}| \to \infty$ but where the expression in Eq. (3.34) is nonvanishing. We will see that this class includes all "topology-changing" gauge transformations.

(II) Finally we lump together all gauge transformations that do not leave A_i^α alone at spatial infinity, i.e., $g(\vec{x})$ does not go to unity as $|\vec{x}| \to \infty$.

For gauge transformations in class (Ia) we can partially integrate the integral in Eq. (3.31) to obtain

$$U = \exp\left[\frac{i}{e}\int d^3x\, \omega^\alpha (D_i E_i)^\alpha\right]. \tag{3.35}$$

By virtue of the constraint condition in Eq. (3.27) we have

$$U|0\rangle = |0\rangle\,, \tag{3.36}$$

and the vacuum is invariant under such transformations. This is just the situation encountered for local gauge transformations in the Abelian theory discussed in Sec. II.

Class (Ib) represents a new situation not occurring in the Abelian theory. Consider a gauge transformation with

$$g(\vec{x}) = (-1)\times\exp\left[\pi i\vec{\sigma}\cdot\frac{\vec{x}}{(\vec{x}^2+\rho^2)^{1/2}}\right], \tag{3.37}$$

where ρ represents an arbitrary scale parameter. As $|\vec{x}|$ goes to infinity $g(\vec{x})$ goes to the identity element of SU(2). An $\omega^\alpha(\vec{x})$ that gives this $g(\vec{x})$ is

$$\omega^\alpha(\vec{x}) = 2\pi\left[\frac{x_\alpha}{(\vec{x}^2+\rho^2)^{1/2}} - \hat{x}_\alpha\right]. \tag{3.38}$$

Note that this has a singularity at the origin. Indeed, topological arguments prove that no continuous $\omega^\alpha(\vec{x})$ that vanishes at infinity can give the $g(\vec{x})$ in Eq. (3.37). This is because three-dimensional space with the point at infinity is topologically equivalent to the surface of a sphere in four dimensions, and Eq. (3.37) represents a homotopically nontrivial mapping of this sphere into the group manifold of SU(2) (also the surface of a four-dimensional sphere).

To treat the singularity at the origin in Eq. (3.38) we consider $g(\vec{x})$ as the product in Eq. (3.37). The factor of (−1) cancels out in the gauge transformation of A_i. We let T represent the unitary operator associated with this transformation; by Eq. (3.31) we have

$$T=\exp\left\{-\frac{2\pi i}{e}\int d^3x\, E_i^\alpha\left[\partial_i\left(\frac{x_\alpha}{(x^2+\rho^2)^{1/2}}\right) - e\epsilon^{\alpha\beta\gamma}A_i^\beta\left(\frac{x_\gamma}{(x^2+\rho^2)^{1/2}}\right)\right]\right\}. \tag{3.39}$$

We refer to this as a topology-changing gauge transformation. All homotopically nonequivalent gauge transformations can be obtained up to a transformation in class (Ia) by taking integer powers of T or its inverse. Changing the scale parameter ρ effectively amounts to making an additional local gauge transformation. As physical states are not affected by such transformations, the action of T on physical states is independent of ρ.

This operator changes the fields by a finite amount over a region of size parametrized by ρ. Thus, it should not take us out of the physical Hilbert space. Also, since T commutes with the Hamiltonian, we expect the vacuum to be an eigenstate of T

$$T|0\rangle=e^{i\theta}|0\rangle, \tag{3.40}$$

where the eigenvalue is of unit magnitude because T is unitary. Since T is a gauge transformation, if we build up our Hilbert space using gauge-invariant operators on the vacuum, all states thus generated will be eigenstates with the same eigenvalue of T. Indeed, Hilbert space will break up into sectors labeled by θ, each built on a vacuum which is the lowest energy eigenstate of the Hamiltonian for a given value of θ. In Sec. VI we will attempt to give a physical interpretation for θ and will present arguments that in three spatial dimensions all the θ vacuums are physically equivalent. Note that this discussion of Hilbert-space sectors labeled by a parameter θ is independent of whether the theory is confined or not.

In Appendix C we show that the quantity

$$q=-\frac{e^2}{24\pi^2}\int d^3x\,\epsilon_{ijk}\mathrm{Tr}[A_i(3\partial_jA_k+2ieA_jA_k)] \tag{3.41}$$

is conjugate to T in the sense that

$$TqT^{-1}=q+1\,. \tag{3.42}$$

This integral represents a coordinate in which the theory is periodic, and T is the symmetry operator that generates translations in this periodic coordinate.

Let us now consider a continuous deformation of the topologically nontrivial $g(\vec{x})$ in Eq. (3.37) to a transformation in class (Ia). The easiest way to do this is to introduce a parameter s that runs from zero to one and consider

$$g(\vec{x},s)=\exp(\pi i s\sigma_3)\times\exp\left\{2\pi i s\,\frac{\sigma^\alpha}{2}\left[\frac{x_\alpha}{(x^2+\rho^2)^{1/2}}\right]\right\}. \tag{3.43}$$

At $s=1$ this is the transformation in Eq. (3.37) while at $s=0$ for all x it is the identity element of SU(2). The important point to note is that for intermediate s, $g(\vec{x},s)$ does not go to the identity element at $|\vec{x}|=\infty$. Thus this continuation of a transformation in class (Ib) to class (Ia) requires that the transformation at an intermediate stage be in class (II). By the topological arguments mentioned earlier, this is a general property of any continuation from class (Ib) to (Ia).

This leads us naturally to a study of class (II) gauge transformations. Based on the results of the previous section, we expect the behavior of the states under these transformations to be rather different in the confined and unconfined phases. Because a class (II) transformation has $g(\vec{x})\neq 1$ at $|\vec{x}|=\infty$, it changes the fields A_i by a nontrivial amount over an infinite volume. In the unconfined phase where A_i is associated with Goldstone vector mesons we expect such a transformation to take us out of the original Hilbert space, as exhibited for the Abelian theory in the last section. This means that in the continuation indicated in Eq. (3.43) the intermediate values of s give an operator that takes us out of the Hilbert space. In the unconfined phase there is no way to continue between operators corresponding to gauge transformations in class (Ia) and (Ib) while staying within the physical Hilbert space. In this case the operator T represents a discrete symmetry much like that encountered by a particle in a periodic potential.

The situation is quite different in the confined phase. Here we do not want massless vector mesons, so, as discussed in the last section, we expect the broken symmetry under transformations in class (II) to be restored. Thus class (Ia) and class (Ib) operators can be continued into one another without leaving the confined Hilbert space. The discrete symmetry becomes a continuous symmetry and the periodic-potential analogy is lost. Indeed, in lattice gauge theory, where the confined phase is the natural one, topology-changing gauge transformations seem to play no important role. In Sec. V we study the above ideas in the Schwinger model.

IV. THE PERTURBATION-THEORY VACUUM

Classically, the lowest-energy configuration for the gauge-theory Hamiltonian in Eq. (3.22) has $F^{\alpha}_{\mu\nu}=0$ everywhere in space. When $F^{\alpha}_{\mu\nu}=0$, A^{α}_i can differ only by a gauge transformation from $A^{\alpha}_i=0$. Because this SU(2) gauge theory has topologically inequivalent gauge transformations, there are topologically inequivalent classical ground states. The Belavin *et al.*[10] pseudoparticle solution represents a finite-action path connecting these inequivalent configurations. In the quantum theory, it has been argued, this provides a path for quantum-mechanical tunneling between the various classical "vacuums."

Of course $F^{\alpha}_{\mu\nu}=0$ does not represent an acceptable quantum state because F^{α}_{0i} and F^{α}_{ij} do not commute. In doing perturbation theory, one starts with a vacuum state that is the true ground state of the theory obtained by setting the coupling constant e equal to zero. In this section we construct this state and study the action of the operator T upon it.

Upon turning off the coupling, the non-Abelian theory of the preceding section becomes a direct product of three independent Abelian gauge theories of the type discussed in Sec. II. Just as in that section we go to momentum space

$$A^{\alpha}_i(\vec{x},t)=\int\frac{d^3k}{(2\pi)^3 2k_0}\tilde{A}^{\alpha}_i(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}\,,$$
$$E^{\alpha}_i(\vec{x},t)=\int\frac{d^3k}{(2\pi)^3 2k_0}\tilde{E}^{\alpha}_i(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}\,, \quad (4.1)$$

and form transverse creation and destruction operators at $t=0$,

$$P_{ij}\tilde{A}^{\alpha}_j(\vec{k},0)=a^{\alpha}_i(\vec{k})+a^{\alpha\dagger}_i(-\vec{k})\,,$$
$$P_{ij}\tilde{E}^{\alpha}_i(\vec{k},0)=-ik_0[a^{\alpha}_i(\vec{k})-a^{\alpha\dagger}_i(-\vec{k})]\,, \quad (4.2)$$

where P_{ij} is the transverse projection operator of Eq. (2.14). The perturbation-theory vacuum $|0\rangle_{\rm PT}$ is defined by

$$a^{\alpha}_i(\vec{k})\,|0\rangle_{\rm PT}=0\,,$$
$$[D_iE_i(\vec{x})]^{\alpha}\,|0\rangle_{\rm PT}=0\,, \quad (4.3)$$

and we normalize

$${}_{\rm PT}\langle 0|0\rangle_{\rm PT}=1\,. \quad (4.4)$$

This state satisfies

$${}_{\rm PT}\langle 0|P_{ij}\tilde{A}^{\alpha}_j|0\rangle_{\rm PT}=0\,. \quad (4.5)$$

It represents an improvement over the classical ground state $A^{\alpha}_i=0$ because some small quantum corrections are included.

We now define the "topologically distinct" perturbation-theory vacuums by the formula

$$|n\rangle_{\rm PT}=T^n|0\rangle_{\rm PT}\,, \quad (4.6)$$

where T is the operator of Eq. (3.39) in the last section and n is any integer. Since T commutes with the Hamiltonian, these states all have the same expectation value for the energy

$${}_{\rm PT}\langle n|H|n\rangle_{\rm PT}={}_{\rm PT}\langle 0|H|0\rangle_{\rm PT}\,. \quad (4.7)$$

Eigenstates of T are easily found from a linear combination of these states

$$|\theta\rangle_{\rm PT}=C\sum_n e^{-in\theta}|n\rangle_{\rm PT}\,,$$
$$T|\theta\rangle_{\rm PT}=e^{i\theta}|\theta\rangle_{\rm PT}\,, \quad (4.8)$$

where C is a normalization factor. Because the true vacuums of the theory should be eigenstates of both T and H as discussed in the last section, these states $|\theta\rangle_{\rm PT}$ should represent a better approximation to the interacting vacuums than the states $|n\rangle_{\rm PT}$.

The state $|n\rangle_{\rm PT}$ differs from the state $|0\rangle_{\rm PT}$ only by a finite gauge transformation in a region of space characterized by the scale parameter ρ. Consequently, the overlap between these states should be nonvanishing. We define

$${}_{\rm PT}\langle 0|1\rangle_{\rm PT}={}_{\rm PT}\langle 0|T|0\rangle_{\rm PT}\equiv\gamma\,. \quad (4.9)$$

Calculation of this overlap is complicated by the nonlinear constraint condition in Eq. (4.3). We do expect the magnitude of γ to be less than unity because T is unitary, and we expect γ to vanish exponentially in e^{-2} as the coupling is taken to zero because the displacement of A^{α}_i by T is of order e^{-1}.

The state $|0\rangle_{\rm PT}$ is an approximation to the vacuum of the unconfined phase of a gauge theory. As such it is not invariant under gauge transformations that do not vanish at spatial infinity. The states $|n\rangle_{\rm PT}$ do not differ from $|0\rangle_{\rm PT}$ outside a region of size ρ and no combination of them can form a state invariant under such gauge transformations. Thus the states $|\theta\rangle_{\rm PT}$ still represent an approximation to the vacuum of the unconfined gauge theory. Unfortunately we conclude that if confinement is to occur in non-Abelian gauge theories, it does not arise in a purely kinematic way from a mixing of the states $|n\rangle_{\rm PT}$.

V. SCHWINGER MODEL

In Sec. II, free Abelian gauge theory was considered as a solvable example of an unconfined theory to show that gauge transformations not vanishing at spatial infinity were not operators in the physical Hilbert space. The Schwinger model provides a solvable confined theory that also possesses topologically nontrivial gauge transformations.[11] Here we discuss the canonical quantization of this model in the temporal gauge and show that gauge transformations that do not vanish at infinity remain operators in the Hilbert space. As a consequence, there is a continuous symmetry connecting the topologically trivial and topology-changing gauge transformations.

The model is defined from the classical Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\bar{\psi}\, \overleftrightarrow{\not\partial}\, \psi + j_\mu A_\mu \ , \tag{5.1}$$

where

$$j_\mu = e\bar{\psi}\gamma_\mu\psi \ , \tag{5.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{5.3}$$

and ψ is a two-component spinor field. The 2×2 Dirac matrices γ_μ satisfy

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \ . \tag{5.4}$$

The electric field is defined by

$$E = F_{01} = -F_{10} \ . \tag{5.5}$$

In the temporal gauge the Lagrangian reduces to

$$\mathcal{L} = \tfrac{1}{2}(\partial_0 A_1)^2 + \tfrac{1}{2}\bar{\psi} i \overleftrightarrow{\not\partial}\psi - j_1 A_1 \ . \tag{5.6}$$

Canonical procedures give the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}E^2 - \tfrac{1}{2}\bar{\psi}\, i\gamma_1 \overleftrightarrow{\nabla}\, \psi + j_1 A_1 \ . \tag{5.7}$$

As in previous sections we work at a fixed time and obtain the quantum theory by imposing the commutation relations

$$\begin{aligned} &[E(x), A_1(y)] = -i\delta(x-y) \ , \\ &[\psi_\alpha(x), \psi_\beta^\dagger(y)]_+ = \delta_{\alpha\beta}\delta(x-y) \ . \end{aligned} \tag{5.8}$$

To avoid ambiguities in defining products of fields at the same point we replace Eq. (5.2) with

$$\begin{aligned} &j_0(x) = e\psi^\dagger(x+\epsilon)\psi(x)e^{i\epsilon A_1(x)} \ , \\ &j_1(x) = e\left[\bar{\psi}(x+\epsilon)\gamma_1\psi(x)e^{-i\epsilon A_1(x)} - \frac{i}{\pi\epsilon}\right], \end{aligned} \tag{5.9}$$

where ϵ is to be taken to zero at the end of any calculation. The constant subtracted from $j_1(x)$ is chosen somewhat arbitrarily by requiring the definition to correspond to normal ordering with respect to a massless free Dirac field. A similar point separation can be used to define the Hamiltonian density.

The equations of motion follow by commuting operators with the Hamiltonian to obtain their time derivatives. This gives

$$\begin{aligned} &\partial_0 A_1 = E \ , \\ &\partial_0 E = -j_1 \ , \\ &\partial_0 j_0 = -\nabla_1 j_1 \ . \end{aligned} \tag{5.10}$$

As in previous sections, Gauss's law does not arise from this procedure since it does not involve time derivatives. Rather, we find

$$\partial_0(\nabla_1 E - j_0) = 0 \ ; \tag{5.11}$$

so, $\nabla_1 E - j_0$ and the Hamiltonian can be simultaneously diagonalized. We impose that any physical state $|\psi\rangle$ satisfy

$$(\nabla_1 E - j_0)\,|\psi\rangle = 0 \ . \tag{5.12}$$

Consequently, for physical states the electric field is determined in terms of $j_0(x)$,

$$E(x)\,|\psi\rangle = \left[E(-\infty) + \int_{-\infty}^{x} j_0(x')dx'\right]|\psi\rangle \ , \tag{5.13}$$

where we have allowed for a possible applied field at spatial infinity. Note that for physical states the connection between j_μ and E takes the simple form

$$j_\mu\,|\psi\rangle = \epsilon_{\mu\nu}\partial_\nu E\,|\psi\rangle \ , \tag{5.14}$$

where $\epsilon_{\mu\nu}$ is antisymmetric with $\epsilon_{01} = 1$.

We now consider the time-independent gauge transformation characterized by the function $\Lambda(x)$

$$\begin{aligned} &A_1 \to UA_1U^{-1} = A_1 + \frac{1}{e}\nabla_1\Lambda \ , \\ &\psi \to U\psi U^{-1} = e^{-i\Lambda(x)}\psi(x) \ . \end{aligned} \tag{5.15}$$

The unitary operator U effecting this transformation is

$$U = \exp\left[\frac{i}{e}\int dx\,(E\nabla_1\Lambda + j_0\Lambda)\right] . \tag{5.16}$$

If $\Lambda(x)$ vanishes at $x = \pm\infty$, then this expression can be partially integrated and as in previous sections the constraint condition requires all physical states to be invariant under this operator. If $\Lambda(x)$ remains finite at infinity, then a surface term survives the partial integration, and we obtain on physical states

$$U\,|\psi\rangle = \exp\left(\frac{i}{e}\{E(-\infty)[\Lambda(\infty) - \Lambda(-\infty)] + Q\Lambda(\infty)\}\right)|\psi\rangle \ , \tag{5.17}$$

where

$$Q = E(\infty) - E(-\infty) = \int dx\, j_0 \tag{5.18}$$

measures the total charge of the state. For a definite charge (usually zero) and a given applied field $E(-\infty)$, this operator just multiplies physical states by a phase.

The above discussion only assumed that $\Lambda(\pm\infty)$ is finite. This agrees with our argument that for a confined theory, symmetry under gauge transformations not vanishing at infinity will be restored. A topology-changing transformation is defined by

$$\Lambda(\infty)-\Lambda(-\infty)=2\pi n\,, \tag{5.19}$$

where n is an integer. We see that such transformations can be continuously deformed into one another by considering noninteger n.

To define a parameter θ as in the previous section, we introduce a particular topology changing transformation with $n=1$,

$$\Lambda(x)=\pi[x/(x^2+\rho^2)^{1/2}-1]\,,$$
$$T=\exp\left[\frac{i}{e}\int dx\,(E\nabla_1\Lambda+j_0\Lambda)\right], \tag{5.20}$$

where ρ is a scale parameter. On physical states we find

$$T|\psi\rangle=e^{i\theta}|\psi\rangle\,, \tag{5.21}$$

where

$$\theta=\frac{2\pi}{e}E(-\infty)\,. \tag{5.22}$$

Thus θ is directly related to the applied field $E(-\infty)$.

The exact solution to this model has been extensively discussed elsewhere.[11] The field $E(x)$ is a free boson field of mass $e/\sqrt{\pi}$. This boson may be thought of as a bound state of a fermion-antifermion pair. The spectrum of the theory is actually independent of θ; however, this is not a general property in two dimensions because in the massive Schwinger model the applied field is not completely shielded and has striking physical consequences.[12] The presence of a fermion mass term does not alter our conclusions on gauge transformations that do not vanish at infinity.

The applied field as measured by θ is not, in general, equal to the average electric field. This is because the vacuum is effectively a dielectric medium and will be polarized by any applied field. For the massless Schwinger model this polarization completely shields the background field, and the average field is zero.[12]

VI. BACKGROUND ELECTRIC FIELDS

In the previous section we saw that for the Schwinger model the phase θ represented a possible applied electric field. In this section we speculate on whether this concept can be extended to the four-dimensional theory discussed in Sec. III. We first study the unconfined phase and then become even more speculative with the confined phase.

If an unconfined phase of the four-dimensional SU(2) gauge theory exists, we might expect the theory to have a classical limit. A space-independent electric field is a solution to the classical gauge-theory equations. We wish to insert such a solution into the expression for T in Eq. (3.39), thus obtaining a classical value for the parameter θ.

To remove any infrared problems we consider space to be restricted to the interior of a sphere of radius $R\gg\rho$. Partially integrating the expression in Eq. (3.39) and using Gauss's law $D_iE_i=0$, we obtain a simple expression for T as a surface integral over this sphere

$$T=\exp\left(-\frac{2\pi i}{e}\int d^2S_i\,E_i^\alpha\hat{r}_\alpha\right), \tag{6.1}$$

where $\hat{r}_\alpha$ is a unit vector in the radial direction. Inserting a constant E_i^α gives

$$T=\exp\left(-i\,\frac{8\pi^2R^2}{3e}\,E_i^\alpha\,\delta_i^\alpha\right) \tag{6.2}$$

or

$$\theta=-\frac{8\pi^2R^2}{3e}\,E_i^\alpha\delta_i^\alpha \tag{6.3}$$

up to a multiple of 2π. To minimize the classical energy density $\frac{1}{2}E_i^\alpha E_i^\alpha$ for a given value of θ we choose

$$E_i^\alpha=E\delta_i^\alpha\,, \tag{6.4}$$

which gives

$$\theta=-\frac{8\pi^2R^2}{e}\,E\,, \tag{6.5}$$

showing that a background electric field can indeed give a classically nonvanishing θ. Note that the solution we are considering is not gauge-invariant. The quantum ground state must involve a superposition of states corresponding to this solution and all local gauge transformations of it.

As R is taken to infinity with θ fixed, the background field E will go to zero as R^{-2}. This property is essential because in an infinite volume a finite uniform field can produce virtual pairs of charged particles and separate them to a large enough distance that their electrostatic energy allows them to become real. In non-Abelian gauge theories there are massless gauge bosons carrying charge e. In a finite volume of dimension R they have an effective mass of order R^{-1}; consequently, for stability, E must go to zero at least as fast as R^{-2}, consistent with Eq. (6.5).

Since θ represents an angle, it can be restricted to the range $-\pi<\theta\leq\pi$. For consistency of our interpretation, θ of order π should correspond to a stable field configuration. Therefore, we need to consider electric fields as large as

$$E=e/8\pi R^2\,. \tag{6.6}$$

Placing a circular parallel-plate capacitor of radius R inside our universe also of radius R, and charging it with $\pm e$ on the plates, we see that whenever the applied external field satisfies

$$E \leq e/2\pi R^2 , \tag{6.7}$$

the plates will be attracted to each other. This suggests that such a configuration is stable. The compatability of Eq. (6.6) with Eq. (6.7) shows no inconsistency in our identification between θ and a background field; however, the puzzling mismatch of the maximum electric field with the field giving $\theta = \pi$ suggests that θ alone is not sufficient to describe the configuration. These quantities match perfectly in two-dimensional electrodynamics.

If our identification of θ as a manifestation of a background electric field is correct, then we conclude that as R goes to infinity, i.e., as the infrared cutoff is removed, measurements probing a finite volume of space will be insensitive to the value of θ. Indeed, θ represents a global property of space and has no microscopic consequences. This is in sharp contrast to the massive Schwinger model, where the allowed background field does not vanish as space is made infinite.

In the confined phase of a gauge theory there should not be any long-range fields corresponding to massless particles. Based on the lattice gauge theories, it has been conjectured that electric fields will form themselves into flux tubes, vortices of finite transverse size carrying an amount of flux as would be produced by a particle in the fundamental representation of the gauge group.[13] For the SU(2) theory considered here, this fundamental unit of charge is $e/2$.

Assuming this formation of flux tubes to be a correct qualitative description of the confined phase, we conjecture that a nonvanishing θ corresponds to a uniform background density of flux tubes. Consider a single such tube parallel to the x_3 axis in our world of radius R and let the electric flux in this tube point in the third isotopic direction. Thus, assume that $E_i^\alpha(r)$ has the form

$$E_i^\alpha(\vec{x}) = \delta_{i3}\delta^{\alpha 3} E(x_1, x_2) . \tag{6.8}$$

Since the net flux has value $e/2$, we require

$$\int dx_1\, dx_2\, E(x_1, x_2) = \frac{e}{2} . \tag{6.9}$$

Inserting this into Eq. (6.1) gives

$$T = \exp\left[-2\pi i \left(1 - \frac{x_1^2 + x_2^2}{R^2}\right)^{1/2}\right] ,$$

where x_1 and x_2 are the coordinates of the center of the flux tube. Thus a density of one flux tube per universe can, depending on its position, give any value of θ. Again we conclude that the value of θ has no microscopic physical consequences.

VII. CONCLUDING REMARKS

We have presented a criterion for confinement in terms of the symmetry properties of the temporal-gauge vacuum under time-independent gauge transformations. From this we conclude that in non-Abelian gauge theories possessing topologically nontrivial gauge transformations the unconfined phase will possess a corresponding discrete symmetry in the physical Hilbert space. In the confined phase this becomes a continuous symmetry. Unfortunately our discussion sheds no light on the existence of an unconfined phase for non-Abelian theories. If this topological vacuum structure is related to confinement, the connection must be more subtle than the essentially kinematic arguments given here.[14]

We have implicitly assumed that the gauge theories under consideration have no more than two phases: confined and unconfined. However, renormalization-group arguments suggest that non-Abelian gauge theories have most likely an odd number of phases.[4] Thus if an unconfined phase exists for such a theory, there are probably still other phases. What characterizes them? Certainly the simplest and most desirable situation is for four-dimensional non-Abelian theories to possess only the confined phase.

We have worked in the temporal gauge. As the canonical quantization of gauge fields is gauge dependent, our conclusions may differ in other gauges. Previous discussions of the photon as a Goldstone boson have been primarily given in the Lorentz gauge $\partial_\mu A_\mu = 0$.[9] This gauge requires an indefinite-metric space containing several parallel and equivalent physical Hilbert spaces. Gauge transformations shift between these equivalent physical spaces; thus, our interpretation of topology-changing transformations as a symmetry of the physical space would undoubtedly change.[15] In an axial gauge such as $A_3 = 0$ the topological structure of the theory must all reside in the properties of A_μ on the boundary of space.[16]

Except with the Schwinger model, we have ignored couplings of fermions to the gauge fields. We feel that if confinement does occur, it should be signaled by the pure gauge sector of the theory. Massless fermions do have dramatic consequences for the tunneling process because, as discussed in Refs. 8 and 17, the tunneling will be accompanied with the creation of zero-momentum fermions. The consequences of this for chiral invariance have been extensively discussed in these references. We have also ignored the fact that e must be renormalized; we feel that ultraviolet problems should be irrelevant to the global structure of the theory.

It would be desirable to have more rigorous arguments than those of the preceding section on the physical relevance of the variable θ. Indeed, if we are wrong and θ has microscopic consequences, then we are faced with the puzzling question as to why such a parameter does not arise naturally in the lattice formulations of gauge theory.[18]

APPENDIX A

In this appendix the behavior of the Abelian vacuum under gauge transformations is analyzed. Consider the transformation

$$A_i \to A_i + \partial_i \Lambda(\vec{x}), \tag{A1}$$

which is generated by the unitary operator

$$U = \exp\left[i \int d^3x\, E_i(\partial_i \Lambda) e^{-\epsilon^2 \vec{x}^2} \right], \tag{A2}$$

where an infrared cutoff has been introduced in the exponent. In order to study its effect on the vacuum, we calculate the expectation value of U, which can be readily reduced to

$$\langle 0|U|0\rangle = \exp\left[-\tfrac{1}{2} \int d^3x d^3y\, \partial_i \Lambda(\vec{x}) \partial_i \Lambda(\vec{y}) \Delta_{ij}(\vec{x}-\vec{y}) e^{-\epsilon^2(\vec{x}^2+\vec{y}^2)} \right], \tag{A3}$$

where

$$\Delta_{ij}(\vec{x}) = \langle 0|E_i(\vec{x})E_j(0)|0\rangle = -(\nabla^2 \delta_{ij} - \nabla_i \nabla_j) I(\vec{x}) \tag{A4}$$

with

$$I(\vec{x}) = \int \frac{d^3k}{(2\pi)^3 2k_0} e^{i\vec{k}\cdot\vec{x}} = \frac{1}{4\pi^2 \vec{x}^2}. \tag{A5}$$

Thus after two integrations by parts,

$$\langle 0|U|0\rangle = \exp\left[-\frac{\epsilon^4}{2\pi^2} \int d^3x\, d^3y \frac{1}{(\vec{x}-\vec{y})^2} (\vec{x}\cdot\vec{y}\delta_{ij} - x_j y_i) \partial_i \Lambda(\vec{x}) \partial_j \Lambda(\vec{y}) e^{-\epsilon^2(\vec{x}^2+\vec{y}^2)} \right]. \tag{A6}$$

For Λ as in Eq. (2.29) this integral can be evaluated exactly and gives the result in Eq. (2.33). Now consider $\Lambda(\vec{x})$ of Eq. (2.36)

$$\Lambda(\vec{x}) = \lambda x_3/(x^2+\rho^2)^{1/2}. \tag{A7}$$

The calculation is somewhat more complicated. Equation (A6) reduces to

$$\langle 0|U|0\rangle = \exp\left[-\frac{\lambda^2\epsilon^4}{3\pi^2} \int d^3x d^3y \frac{\vec{x}\cdot\vec{y}}{(\vec{x}-\vec{y})^2 (x^2+\rho^2)^{1/2} (y^2+\rho^2)^{1/2}} e^{-\epsilon^2(\vec{x}^2+\vec{y}^2)} \right]. \tag{A8}$$

Scaling x and y by a factor of ϵ and doing the angular integrations yields

$$\langle 0|U|0\rangle = \exp\left\{ -\frac{2\lambda^2}{3} \int_0^\infty dx\, dy\, e^{-x^2-y^2} \frac{1}{[x^2+(\rho\epsilon)^2]^{1/2}} \frac{1}{[y^2+(\rho\epsilon)^2]^{1/2}} \left[xy(x^2+y^2) \ln\left(\frac{(x-y)^2}{(x+y)^2} \right) - 4x^2y^2 \right] \right\}. \tag{A9}$$

In the limit of vanishing ϵ the integral remains well defined and can be done numerically to give

$$\langle 0|U|0\rangle = \exp(-\lambda^2 \times 0.55\ldots). \tag{A10}$$

In general, by dimensional arguments, if

$$\Lambda(x) \underset{|x|\to\infty}{\sim} |x|^P$$

then

$$\langle 0|U|0\rangle = \exp(-C\epsilon^{2P}), \tag{A11}$$

where C is some positive constant.

APPENDIX B

Here we derive the operator U in Eq. (3.31) that implements SU(2) gauge transformations. Introduce a parameter s that runs from 0 to 1 consider

$$g(\vec{x}, s) = \exp\left[is \frac{\vec{\sigma}}{2} \cdot \vec{\omega}(\vec{x}) \right]. \tag{B1}$$

We want a $U(s)$ satisfying

$$U(s) A_i U^{-1}(s) = g(s) A_i g^{-1}(s) + \frac{i}{e} [\partial_i g(s)] g^{-1}(s). \tag{B2}$$

We look for U of the form

$$U(s)=e^{isQ}, \tag{B3}$$

where Q is to be determined. Differentiating Eq. (B2) with respect to s gives

$$[iQ, UA_iU^{-1}]=g\left[\frac{i\vec{\sigma}}{2}\cdot\vec{\omega}, A_i\right]g^{-1}+\frac{i}{e}\frac{d}{ds}\{[\partial_i g(s)]g^{-1}(s)\}$$
$$=g[iQ, A_i]g^{-1}. \tag{B4}$$

This can be rewritten

$$[Q,A_i]=\left[\frac{\vec{\sigma}\cdot\vec{\omega}}{2}, A_i\right]+\frac{i}{e}\left(g^{-1}\left\{\frac{d}{ds}[(\partial_i g)g^{-1}]\right\}g\right). \tag{B5}$$

The last term is worked out by noting that for any function $f(s)$

$$\frac{d}{ds}f(s)=\frac{d}{d\epsilon}f(s+\epsilon)\Big|_{\epsilon=0} \tag{B6}$$

This gives

$$\frac{d}{ds}\{[\partial_i g(s)]g^{-1}(s)\}=\frac{d}{d\epsilon}\{\partial_i[g(s)g(\epsilon)]g^{-1}(\epsilon)g^{-1}(s)\}\Big|_{\epsilon=0}$$
$$=g(s)\frac{d}{d\epsilon}\{[\partial_i g(\epsilon)]g^{-1}(\epsilon)\}g^{-1}(s)\Big|_{\epsilon=0}$$
$$=g(s)i\frac{\vec{\sigma}}{2}\cdot\partial_i\vec{\omega}g^{-1}(s). \tag{B7}$$

Equation (B5) now becomes

$$[Q,A_i]=\left[\frac{\vec{\sigma}\cdot\vec{\omega}}{2}, A_i\right]+\frac{i}{e}\frac{\vec{\sigma}}{2}\cdot\partial_i\omega, \tag{B8}$$

which is equivalent to

$$[Q,A_i^\alpha]=\frac{i}{e}(\partial_i\omega^\alpha-e\epsilon^{\alpha\beta\gamma}A_i^\beta\omega^\gamma). \tag{B9}$$

Although ω^α is not a dynamical field, we formally write this in the simpler form

$$[Q,A_i^\alpha]=\frac{i}{e}(D_i\omega)^\alpha. \tag{B10}$$

We require that U also generate the correct gauge transformation on E_i^α. A similar argument to the above implies

$$[Q,E_i^\alpha]=i\epsilon^{\alpha\beta\gamma}\omega^\beta E_i^\gamma \tag{B11}$$

From the canonical commutation relations we find an operator that yields Eqs. (B10) and (B11)

$$Q=-\frac{1}{e}\int d^3x\,E_i^\alpha(D_i\omega)^\alpha. \tag{B12}$$

This is unique up to an irrelevant arbitrary constant that we drop. Thus we find

$$U(s)=\exp\left[-\frac{is}{e}\int d^3x\,E_i^\alpha(D_i\omega)^\alpha\right]. \tag{B13}$$

Setting s to unity gives Eq. (3.31).

APPENDIX C

Here we construct a coordinate q conjugate to the topology-changing transformation in Eq. (3.39). The number of times an arbitrary mapping $g(\vec{x})$ covers the group SU(2) is given by[10] Eq. (3.34). For the $g(\vec{x})$ in Eq. (3.37) this integral has the value one. We wish to construct out of the dynamical variables A_i an operator q with the property

$$TqT^{-1}=q+1. \tag{C1}$$

Using

$$TA_iT^{-1}=gA_iq^{-1}+\frac{i}{e}\partial_i gg^{-1}, \tag{C2}$$

we see

$$T[\epsilon_{ijk}\mathrm{Tr}(A_iA_jA_k)]T^{-1}=\epsilon_{ijk}\mathrm{Tr}\left[A_iA_jA_k+3\frac{i}{e}A_iA_jg^{-1}\partial_kg+3\left(\frac{i}{e}\right)^2A_ig^{-1}\partial_jgg^{-1}\partial_kg+\left(\frac{i}{e}\right)^3\partial_igg^{-1}\partial_jgg^{-1}\partial_kgg^{-1}\right] \tag{C3}$$

and

$$T[\epsilon_{ijk}\mathrm{Tr}(A_i\partial_jA_k)]T^{-1}=\epsilon_{ijk}\mathrm{Tr}\left[A_i\partial_jA_k+2A_iA_jg^{-1}\partial_kg+3\frac{i}{e}A_ig^{-1}\partial_jgg^{-1}\partial_kg+\frac{i}{e}\partial_iA_jg^{-1}\partial_kg+\left(\frac{i}{e}\right)^2\partial_igg^{-1}\partial_jgg^{-1}\partial_kgg^{-1}\right]. \tag{C4}$$

Taking the appropriate combination of Eqs. (C3) and (C4) to cancel the cross terms yields

$$T\left[\epsilon_{ijk}\mathrm{Tr}\left(2A_iA_jA_k-3\frac{i}{e}A_i\partial_jA_k\right)\right]T^{-1}=\epsilon_{ijk}\mathrm{Tr}\left(2A_iA_jA_k-3\frac{i}{e}A_i\partial_jA_k\right)-\left(\frac{i}{e}\right)^3\epsilon_{ijk}\mathrm{Tr}(\partial_igg^{-1}\partial_jgg^{-1}\partial_kgg^{-1})$$
$$-3\left(\frac{i}{e}\right)^2\partial_i[\epsilon_{ijk}\mathrm{Tr}(A_jg^{-1}\partial_kg)]. \tag{C5}$$

Integrating over space and using Eq. (3.34), we obtain

$$T\left[\left(-\frac{ie^3}{24\pi^2}\right)\int d^3x\,\epsilon_{ijk}\,\mathrm{Tr}\left(2A_iA_jA_k-3\,\frac{i}{e}A_i\partial_jA_k\right)\right]T^{-1}$$

$$=-\frac{ie^3}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{Tr}\left(2A_i\,A_jA_k-3\,\frac{i}{e}A_i\partial_jA_k\right)+1\,. \quad \text{(C6)}$$

So

$$q=-\frac{e^2}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{Tr}A_i\left(3\partial_jA_k-\frac{e}{i}\;2A_jA_k\right) \quad \text{(C7)}$$

is the coordinate being sought.

---

*Work supported by U. S. Energy Research and Development Administration.

[1]H. D. Politzer, Phys. Rep. 14C, 129 (1974); W. Marciano and H. Pagels, Rockefeller University Report No. COO-2232B-130 (unpublished).

[2]T. Appelquist, J. Carazzone, H. Kluberg-Stern, and M. Roth, Phys. Rev. Lett. 36, 768 (1976); 36, 1161(E) (1976); Y. P. Yao, *ibid.* 36, 653 (1976); T. Appelquist, M. Dine, and I. Muzinich, Phys. Lett. 69B, 231 (1977).

[3]K. Wilson, Phys. Rev. D 10, 2445 (1974); in *New Phenomena in Subnuclear Physics*, proceedings of the 14th Course of the International School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977); J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); T. Banks *et al.*, *ibid.* 15, 1111 (1977).

[4]L. P. Kadanoff, Rev. Mod. Phys. 49, 267 (1977).

[5]R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D 10, 3376 (1974); 11, 2098 (1975); 11, 2104 (1975).

[6]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 810 (1975) [Sov. Phys.—JETP 42, 413 (1975)]; 69, 1457 (1975) [42, 743 (1975)]; L. P. Kadanoff, Ann. Phys. (N.Y.) 100, 359 (1976).

[7]A. M. Polyakov, Phys. Lett. 59B, 82 (1975); Nucl. Phys. B120, 429 (1977).

[8]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); J. L. Gervais and B. Sakita, Phys. Rev. D (to be published).

[9]R. A. Brandt and W.-C. Ng, Phys. Rev. D 10, 4198 (1974); R. Ferrari and L. E. Picasso, Nucl. Phys. B31, 316 (1971); E. A. Ivanov and V. I. Ogievetsky, Zh. Eksp. Teor. Fiz. Pis'ma Red. 23, 661 (1976) [JETP Lett. 23, 606 (1976)]; G. S. Guralnik, Phys. Rev. Lett. 13, 295 (1964).

[10]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, Phys. Lett. 59B, 85 (1975).

[11]J. Schwinger, Phys. Rev. 128, 2425 (1962); J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971); J. Kogut and L. Susskind, Phys. Rev. D 10, 3468 (1974).

[12]S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) 93, 267 (1975); S. Coleman, *ibid.* 101, 239 (1976); E. H. Monsay and T. N. Tudron, Phys. Rev. D (to be published).

[13]J. Kogut and L. Susskind, Phys. Rev. D 9, 3501 (1974).

[14]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 66B, 375 (1977).

[15]Z. F. Ezawa and D. E. L. Pottinger, Phys. Lett. 66B, 45 (1977); 66B, 445 (1977).

[16]S. Wadia and T. Yoneya, Phys. Lett. 66B, 341 (1977).

[17]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976); R. D. Peccei and H. R. Quinn, *ibid.* 16, 1791 (1977).

[18]M. Lüscher, DESY Report No. 77/16 (unpublished).