$$\frac{1}{\sqrt{2}}c_{\text{GOR}} = 2\frac{m_\pi{}^2 - m_K{}^2}{m_\pi{}^2 + 2m_K{}^2} \simeq -\left[1 - \frac{3}{2}\left(\frac{m_\pi}{m_K}\right)^2\right] = -0.88\ .$$

For $l_2 = -\frac{1}{2}m_\pi{}^2$, we get

$$\frac{c}{\sqrt{2}} = \frac{19m_\pi{}^2 - 16m_K{}^2}{11m_\pi{}^2 + 16m_K{}^2} \simeq -\left[1 - \frac{15}{8}\left(\frac{m_\pi}{m_K}\right)^2\right] = -0.85\ .$$

The recalculation of c according to (29) is only slightly influenced by introducing the dependence on l_2.

Our purpose has been to provide an example of symmetry breaking which admits $l_2 \neq 0$. The theoretical result from $\pi\pi$ scattering,[6] $l_2 \simeq -\frac{1}{2}m_\pi{}^2$, has been given a quantitative interpretation in terms of small departures from the case $l_2 = 0$. It remains to be seen whether similar effects in other processes are too small to provide further tests of Eq. (24).

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[1]Latin superscripts are used to denote isovector indices; repeated, these are summed from 1 to 3. Greek superscripts, appearing later, are used to denote octet indices; repeated, these are summed from 1 to 8.

[2]The Clebsch-Gordan coefficients ξ_{Jjk} are symmetric in jk and satisfy $\xi_{Jjj}=0$, $\xi_{Jjk}\xi_{Kjk}=\delta_{JK}$, and, summing $J=1$ to 5,

$$\xi_{Jjk}\xi_{Jlm} = \tfrac{1}{2}(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) - \tfrac{1}{3}\delta_{jk}\delta_{lm}.$$

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Positivity, Subtractions, and the Moduli of Scattering Amplitudes*

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We show that if an amplitude has certain positivity properties as would be given by an optical theorem and if it obeys a dispersion relation with a known number of subtractions, then the amplitude can be uniquely and explicitly found from its modulus on the cuts and a known number of additional parameters. Applying this result to forward polarized Compton scattering, we find that the necessary additional parameters are fixed by low-energy theorems in terms of the static electromagnetic properties of the target.

Analyticity has proven to be one of the most useful tools of recent high-energy physics. With varying degrees of rigor, theorists have shown scattering amplitudes to be analytic functions of particle energies except for singularities which are often related to other physical processes. Dispersion relations then correlate these processes via contour integrals.

Recent interest has appeared on the constraints analyticity imposes on amplitudes whose modulus is known on the branch cuts in the complex plane.[1-4] In general such constraints take the form of inequalities. Okubo[4] has recently summarized several applications of these constraints and briefly mentioned some conditions under which they become equalities. One particularly interesting result of experimental relevance is a set of bounds on the pion electromagnetic radius in terms of the experimentally measured modulus of the pion electromagnetic form factor on its cut.[1,3]

In this paper we wish to add some more input and thus show that certain amplitudes can be explicitly constructed from their magnitude on the cuts and a small number of additional parameters. We study forward scattering amplitudes, which satisfy certain positivity conditions, and we impose subtraction assumptions on the usual dispersion relations. We will finally apply our results to the case of Compton scattering from a spin-$\frac{1}{2}$ target. In this case low-energy theorems uniquely fix the additional parameters needed to determine the amplitudes. We demonstrate that both independent amplitudes are uniquely given in terms of the polarized forward differential cross sections above the inelastic threshold and the static electromagnetic properties of the target.

The essential reason that we can obtain these stronger results is that positivity coupled with subtraction assumptions strongly limits the number of zeros of the amplitude in the complex plane. Uncertainties in the positions and number of zeros limit the determination of an amplitude from its modulus on its cuts. However, Jin and Martin[5] have shown that the number of zeros can be limited. Thus to locate them requires only a limited amount of additional information. In the case of Compton scattering, this additional information comes from low-energy theorems.

Let us begin by listing the assumptions that we impose on our amplitude. We consider a function $f(\nu)$ which is a real analytic function in the complex ν plane except for cuts from ν_0 to ∞ and $-\infty$ to $-\nu_0$. We do not allow $f(\nu)$ to have essential singularities on the cuts. We make the assumption that $f(\nu)$ satisfies a twice-subtracted dispersion relation

$$f(\nu)=f(0)+\nu f'(0)+\frac{\nu^2}{\pi}\int_{-\infty}^{\infty}\frac{d\nu'\,\mathrm{Im}f(\nu'+i\epsilon)}{\nu'^2(\nu'-\nu)}. \tag{1}$$

We will discuss later the case of a different number of subtractions.

The only other assumption which we shall need is positivity. We assume that we are discussing a forward elastic scattering amplitude with an optical theorem

$$\mathrm{Im}f(\nu+i\epsilon)\propto\sigma_T(\nu)\geqslant 0 \quad\text{for}\quad \nu>\nu_0, \tag{2}$$

where $\sigma_T(\nu)$ is the total cross section for the particles initiating the reaction described by $f(\nu)$. The proportionality constant implied by Eq. (2) can in general be taken to be positive so that

$$\mathrm{Im}f(\nu+i\epsilon)\geqslant 0 \quad\text{for}\quad \nu>\nu_0. \tag{3}$$

Now if we take $\nu+i\epsilon$ to $-\nu-i\epsilon$ we obtain the elastic amplitude for the antiparticle of the original beam on the same target. This also obeys an optical theorem implying

$$\mathrm{Im}f(-\nu-i\epsilon)\geqslant 0 \quad\text{for}\quad \nu>\nu_0, \tag{4}$$

or, since $f(\nu)$ is a real analytic function,

$$\mathrm{Im}f(\nu+i\epsilon)\leqslant 0 \quad\text{for}\quad \nu<-\nu_0. \tag{5}$$

Equations (3) and (5) represent the constraint of positivity that we assume $f(\nu)$ satisfies.

Having presented our assumptions on $f(\nu)$, we now attack the problem of what can be said about the amplitude from knowledge of its modulus on the cuts. We therefore take as given $|f(\nu)|$ for $\nu\geqslant\nu_0$ and $\nu\leqslant-\nu_0$. To proceed we introduce the auxiliary function $f_0(\nu)$ defined explicitly in terms of $|f(\nu)|$ on the cuts by

$$f_0(\nu)=\exp\left(\frac{(\nu_0^2-\nu^2)^{1/2}}{\pi}\times\int_{-\infty}^{\infty}\frac{d\nu'\,\theta(\nu^2-\nu_0^2)\epsilon(\nu')}{(\nu'^2-\nu_0^2)^{1/2}(\nu'-\nu)}\ln|f(\nu')|\right), \tag{6}$$

where

$$\theta(x)=\begin{cases}1, & x\geqslant 0\\ 0, & x<0,\end{cases}$$
$$\epsilon(x)=2\theta(x)-1=\begin{cases}1, & x\geqslant 0\\ -1, & x<0,\end{cases} \tag{7}$$

and we choose the branch of the square-root function with positive real part. Note that the integral in Eq. (6) is convergent since $f(\nu)$ is polynomially bounded and because we have assumed it has no essential singularities on the cut.

The function $f_0(\nu)$ has several interesting properties which may be verified directly from the definition. First, it is a real analytic function in the ν plane except for cuts from ν_0 to ∞ and $-\infty$ to $-\nu_0$; this is the same cut structure as possessed by $f(\nu)$. Second, the modulus of $f_0(\nu)$ equals the modulus of $f(\nu)$ on these cuts. Third, $f_0(\nu)$ has no zeros in the complex plane away from the cuts. It is this third condition that assures us that $f_0(\nu)$ need not be identical with $f(\nu)$.

In general $f(\nu)$ may have zeros in the complex ν plane. For every ν_i which represents a zero of $f(\nu)$ away from its cuts, we define a function

$$Z(\nu,\nu_i)=\frac{(\nu_0-\nu_i)^{1/2}(\nu_0+\nu)^{1/2}-(\nu_0+\nu_i)^{1/2}(\nu_0-\nu)^{1/2}}{(\nu_0-\nu_i)^{1/2}(\nu_0+\nu)^{1/2}+(\nu_0+\nu_i)^{1/2}(\nu_0-\nu)^{1/2}}. \tag{8}$$

We pick the value of the square-root function with positive real part. If $-\nu_0\leqslant\nu_i\leqslant\nu_0$, then $Z(\nu,\nu_i)$ is a real analytic function in the cut ν plane with a zero at $\nu=\nu_i$ and $|Z(\nu,\nu_i)|=1$ on the cuts. If ν_i is complex we know that since $f(\nu)$ is a real analytic function it has another zero at ν_i^*. In this case the product $Z(\nu,\nu_i)Z(\nu,\nu_i^*)$ has zeros at ν_i and ν_i^*, is a real analytic function in the cut ν plane, and has unit magnitude on the cuts.

Removing a factor of $Z(\nu, \nu_i)$ for each zero $f(\nu)$, we define a function $D(\nu)$ by

$$f(\nu) = D(\nu)\left(\prod_i Z(\nu, \nu_i)\right) f_0(\nu). \tag{9}$$

The function $D(\nu)$ has the following properties: (a) it is analytic in the cut ν plane, (b) it has no zeros, (c) it has unit magnitude on the cuts, (d) it has no essential singularities on the cuts, and (e) it is polynomially bounded at infinity. These conditions restrict $D(\nu)$ to be of the form

$$D(\nu) = \pm\exp[-K(\nu_0^2 - \nu^2)^{1/2}], \tag{10}$$

where $K \geq 0$. Without using positivity, we have now found the most general form for $f(\nu)$ in terms of its modulus on the cuts:

$$f(\nu) = \pm\left(\prod_i Z(\nu, \nu_i)\right) f_0(\nu) \exp[-K(\nu_0^2 - \nu^2)^{1/2}]. \tag{11}$$

Let us now add our assumption of positivity. The first observation is that if K does not vanish in Eq. (11), then $f(\nu)$ falls faster than any power of ν as $\nu \to i\infty$. It can be readily verified from the original dispersion relation that positivity rules out such a behavior.[5,6] We can thus safely set $K=0$. This result is independent of the number of subtractions in the dispersion relation.

Jin and Martin[5] have discussed the constraints that positivity imposes on the number of zeros in an amplitude. We will rederive here the result that positivity coupled with the assumption of only two subtractions allows $f(\nu)$ to have at *most* two zeros. Once we know that $f(\nu)$ can only have two zeros, two additional statements about $f(\nu)$ will fix their positions.

To demonstrate that $f(\nu)$ has at most two zeros away from the cuts, we temporarily assume that there are at least three and we will show a contradiction. If there are three zeros, we can write a dispersion relation for $f(\nu)$ subtracted at these points and obtain

$$f(\nu) = \frac{(\nu-\nu_1)(\nu-\nu_2)(\nu-\nu_3)}{\pi} \times \int_{\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu'+i\epsilon)}{(\nu'-\nu)(\nu'-\nu_1)(\nu'-\nu_2)(\nu'-\nu_3)}. \tag{12}$$

Taking ν to infinity we find

$$f(\nu) \sim -\frac{\nu^2}{\pi} \int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu')}{(\nu'-\nu_1)(\nu'-\nu_2)(\nu'-\nu_3)}. \tag{13}$$

A ν^2 behavior at infinity would not allow an only twice-subtracted dispersion relation as assumed in Eq. (1); therefore, we must have

$$\int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu'+i\epsilon)}{(\nu'-\nu_1)(\nu'-\nu_2)(\nu'-\nu_3)} = 0. \tag{14}$$

If all three roots lie on the real axis between $-\nu_0$ and $+\nu_0$, the integrand in Eq. (14) is positive so the equation is impossible. This means that at least one of the three zeros, take it to be ν_1, must be complex. If it is complex, another zero occurs at ν_1^*. Take $\nu_2 = \nu_1^*$ and rewrite Eq. (14):

$$\int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu')}{|\nu'-\nu_1|^2 |\nu'-\nu_3|^2} (\nu' - \nu_3^*) = 0. \tag{15}$$

If ν_3 is real and between $-\nu_0$ and ν_0 the integrand is positive and again we have a contradiction. If ν_3 is complex, by taking combinations of the real and imaginary part of (15) we obtain

$$\int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu')\nu'}{|\nu'-\nu_1|^2 |\nu'-\nu_3|^2} = 0. \tag{16}$$

Again this is a positive quantity so we have a contradiction. The result of this is that $f(\nu)$ can have at most two zeros away from its cuts. We will discuss later what happens when the number of subtractions is changed.

Given that $f(\nu)$ has at most two zeros, we can write

$$f(\nu) = \pm Z(\nu, \nu_1) Z(\nu, \nu_2) f_0(\nu). \tag{17}$$

Note that the case where $f(\nu)$ has one or no zeros is actually included in Eq. (17) if we allow ν_i to go to $\pm\nu_0$ since $Z(\nu, \pm\nu_0) = \mp 1$.

We can now show that the positive sign must be taken in Eq. (17). To do this first write a dispersion relation for $f(\nu)$ subtracted at ν_1 and ν_2:

$$f(\nu) = \frac{(\nu-\nu_1)(\nu-\nu_2)}{\pi} \times \int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu'+i\epsilon)}{(\nu'-\nu)(\nu'-\nu_1)(\nu'-\nu_2)}. \tag{18}$$

This implies

$$f(0) = \frac{\nu_1\nu_2}{\pi} \int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im} f(\nu')}{\nu'(\nu'-\nu_1)(\nu'-\nu_2)}. \tag{19}$$

Now for either complex roots or real roots between $-\nu_0$ and ν_0 we have

$$(\nu'-\nu_1)(\nu'-\nu_2) \geq 0$$

on the cuts. This means that $f(0)$ has the same sign as $\nu_1\nu_2$. Observing that $Z(0, \nu_1)Z(0, \nu_2)$ has the same sign as $\nu_1\nu_2$, and noting from the definition that $f_0(0) > 0$, we see that in Eq. (17) we must always choose the positive sign:

$$f(\nu) = Z(\nu, \nu_1) Z(\nu, \nu_2) f_0(\nu). \tag{20}$$

If $f(0)$ happens to vanish, this argument must be modified slightly; however, the result remains the same.

We now have an explicit expression for $f(\nu)$ in terms of its modulus on the cuts and two additional parameters. These parameters could be determined by additional information on $f(\nu)$. For example, if $f(0)$ and $f'(0)$ are both known, we can solve for ν_1 and ν_2. Elementary algebra gives the following result:

$$\nu_i = -\nu_0\left(\frac{2\nu_0(f'f_0 - f_0'f)(f_0^2 - f^2) \pm 2(f_0 - f)^2[\nu_0^2(f'f_0 - f_0'f)^2 - ff_0(f_0 - f)^2]^{1/2}}{(f_0 - f)^4 + 4\nu_0^2(f'f_0 - f_0'f)^2}\right), \tag{21}$$

where $f = f(0)$, $f' = f'(0)$, $f_0 = f_0(0)$, and $f_0' = f_0'(0)$. The positive sign in front of the square-root term gives one zero while the minus sign gives the other.

By combining Eqs. (20), (21), (6), and (8), we clearly obtain an explicit expression for $f(\nu)$ in terms of its modulus on the cuts and the subtraction constants of the dispersion relation. This is, of course, different input than used in the usual dispersion relation, which determines the amplitude in terms of its imaginary part on the cuts and the subtraction constants. We must emphasize, however, that positivity was crucial to our obtaining a unique result.

We now list several comments on our result and its applications.

(i) The number of subtractions in the dispersion relation is closely related to the number of zeros that the function may have. The general result of Jin and Martin[5] is that if there are $2n$ or $2n-1$ subtractions, the amplitude can have at most $2n$ zeros. Note that an unsubtracted dispersion relation allows for no zeros at all away from the cuts; consequently, an unsubtracted amplitude satisfying positivity is uniquely determined by its modulus on the cuts.

(ii) In many applications of physical interest, such as pion-nucleon scattering, the amplitudes have poles on the real axis between the cuts. Such poles can be accommodated in this discussion by replacing $f(\nu)$ by $\prod_i(\nu - \nu_i)f(\nu)$, where ν_i is the position of the i'th pole. It is important to note, however, that $\prod_i(\nu - \nu_i)f(\nu)$ may require more subtractions than $f(\nu)$ and therefore more zeros are allowed. In general, each additional pole allows one more zero to be present. Knowledge of the residue of the pole provides an additional constraint on the locations of zeros. Note that for an odd number of poles the discussion of positivity on the left-hand cut is modified.

(iii) If $f(\nu)$ is known to be an even function of ν, then $f'(0) = 0$ automatically and we need only know $f(0)$ in addition to the modulus on the cuts in order to determine $f(\nu)$. An even function has at most a pair of zeros at $\pm\nu_1$. Okubo[4] has pointed out that if $f(\nu_0)$ is negative and $f(\nu)$ is even, then it can have no zeros at all if it satisfies our other assumptions. In this case it is uniquely determined by only its modulus on the cuts.

(iv) There are constraints which $f(0)$ and $f'(0)$ must satisfy in order for a solution for $f(\nu)$ to exist at all. These constraints must be satisfied independent of the positivity assumption. They are an obvious generalization of the bounds of Refs. 1, 3, and 4 to functions with two cuts. In the notation of Eq. (21), we must have

$$|f(0)| \leq f_0(0) \tag{22}$$

and

$$\frac{f_0'}{f_0} + \frac{f_0^2 - f^2}{2\nu_0|f|f_0} \geq \frac{f'}{f} \geq \frac{f_0'}{f_0} - \frac{f_0^2 - f^2}{2\nu_0|f|f_0}. \tag{23}$$

Note that if $f_0 = |f|$, then f' is uniquely determined. Let us note that both equations (22) and (23) must remain true if f_0 and f_0' are calculated from an upper bound to $|f(\nu)|$ rather than $|f(\nu)|$ itself.

(v) Although we have used positivity, Eq. (20) does not manifestly exhibit it. For this reason we feel that this constraint has not been exhausted. Indeed, positivity should impose further conditions on the allowed behavior of $|f(\nu)|$ on its cuts. We will discuss this no further here.

(vi) As we have stated the problem, it is directly applicable to forward Compton scattering. Following Damashek and Gilman,[7] we note that there are two independent amplitudes describing forward Compton scattering from a spin-$\frac{1}{2}$ target. Calling the laboratory photon energy ν, we write the amplitude for scattering off the target with both photon and target spins parallel to each other and parallel to the beam direction as

$$f_p(\nu) = f_1(\nu) - f_2(\nu), \tag{24}$$

while when the spins are antiparallel to each other we have

$$f_a(\nu) = f_1(\nu) + f_2(\nu). \tag{25}$$

Our normalization is such that

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab},0^\circ} = |f(\nu)|^2. \tag{26}$$

We work here to lowest nonvanishing order in the electromagnetic charge. Both $f_1(\nu)$ and $f_2(\nu)$ can be continued as analytic functions into the complex ν plane, $f_1(\nu)$ being an even function of ν and $f_2(\nu)$ odd. Thus $f_p(\nu)$ and $f_a(\nu)$ can also be continued so that $f_p(-\nu)=f_a(\nu)$. Positivity tells us that $\operatorname{Im} f_p(\nu+i\epsilon)\geqslant 0$ and $\operatorname{Im} f_a(\nu+i\epsilon)\geqslant 0$ for $\nu \geqslant \nu_0$ = inelastic threshold. Usual Regge theory tells us that $f_p(\nu)$ satisfies a twice-subtracted dispersion relation. All of this discussion indicates that $f_p(\nu)$ satisfies all the conditions we imposed on $f(\nu)$ at the beginning of this paper. Therefore $f_p(\nu)$ can be constructed from $|f_p(\nu)|$ on the cuts, $f_p(0)$, and $f_p'(0)$. Of course, Eq. (26) gives $|f_p(\nu)|$ on both cuts in terms of differential cross sections for polarized photons on a polarized target above the inelastic threshold.

A remarkable fact about Compton scattering is that $f_p(0)$ and $f_p'(0)$ are both determined by low-energy theorems,

$$f_p(0)=f_1(0) = -\frac{\alpha Q^2}{M}, \tag{27}$$

$$f_p'(0)=-f_2'(0) = +\frac{\alpha K^2}{2M^2}, \tag{28}$$

where $\alpha=\frac{1}{137}$, Q is the target charge in units of the electron charge, K is the target anomalous magnetic moment, and M is the target mass. Thus our result is that $f_p(\nu)$, and therefore both $f_1(\nu)$ and $f_2(\nu)$, can be explicitly constructed from knowledge of polarized differential cross sections and static electromagnetic properties of the target.

We remark here that knowledge of $|f_1(\nu)|$ on the cuts determines $f_1(\nu)$. This amplitude satisfies our assumptions and $f_1(0)$ is given in Eq. (27) while $f_1'(0)=0$ because $f_1(\nu)$ is even. However, knowledge of $|f_2(\nu)|$ on the cuts is *not* sufficient to determine $f_2(\nu)$. This is because $f_2(\nu)$ does not satisfy a positivity constraint. Indeed there is no theoretical limit on the number of zeros in $f_2(\nu)$.

(vii) The utility of our result for the experimental determination of amplitudes is not entirely clear. In the case of Compton scattering, the amplitude $f_1(\nu)$ can be quite easily found from unpolarized total cross-section data through the use of the usual dispersion relation.[7] Our result does provide a method of determining $f_2(\nu)$ from polarized differential cross-section data. One could also find $f_2(\nu)$ from conventional dispersion relations using measurements of polarized total cross sections. Either the total or the differential cross-section experiments needed to find $f_2(\nu)$ will require both a polarized photon beam and a polarized target, although in neither case are final-state polarization measurements needed. We suspect that the total cross-section measurement will be simpler; consequently, our result will probably only serve as a check on the usual dispersion relation calculation of $f_2(\nu)$. Since in our general discussion we have restricted ourselves to amplitudes satisfying positivity, a total cross-section measurement coupled with a dispersion relation will always compete with our method for determining an amplitude.

In summary, we have reviewed the problem of constructing an amplitude given its modulus on its cuts. Then we have studied what results from the additional assumptions of positivity and a definite number of subtractions in the usual dispersion relations. In the case of two subtractions we have explicitly demonstrated that the amplitude is uniquely determined from its modulus on the cuts and knowledge of the subtraction constants. We point out that in the case of Compton scattering these subtraction constants are determined by the static electromagnetic properties of the target.

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