Minimally doubled chiral fermions

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Chiral symmetry crucial to our understanding of hadronic physics

- pions are waves on a background quark condensate $\langle \bar{\psi}\psi \rangle$
- chiral extrapolations essential to practical lattice calculations

Anomaly removes classical $U(1)$ chiral symmetry

- $SU(N_f) \times SU(N_f) \times U_B(1)$
- non trivial symmetry requires $N_f \geq 2$
On the lattice ignoring the anomaly gives doublers

- naive fermions: 16 species, exact $U(4)_L \times U(4)_R$ symmetry
- staggered fermions: 4 species (tastes), one exact chiral symmetry
- Wilson fermions: one light species
  - all chiral symmetries broken by doubler mass term
- overlap, domain wall, perfect actions: $N_f$ arbitrary but
  - not ultra-local: computationally intensive
- anomaly hidden, $\gamma_5 \neq \hat{\gamma}_5$, $\text{Tr} \hat{\gamma}_5 = 2\nu \neq 0$
Minimally doubled chiral fermion actions have just 2 species

- Karsten 1981
- Wilczek 1987
- recent revival: MC, Borici, Bedaque Buchoff Tiburzi Walker-Loud

Motivations

- failure of rooting for staggered
- lack of chiral symmetry for Wilson
- computational demands of overlap, domain-wall approaches

Elegant connection to the electronic structure of graphene

- vanishing mass protected by topological considerations
Graphene: two dimensional hexagonal lattice of carbon atoms

Held together by strong “sigma” bonds, $sp^2$

One “pi” electron per site can hop around

Consider only nearest neighbor hopping in the pi system
  • tight binding approximation

- http://online.kitp.ucsb.edu/online/bblunch/castroneto/
Fortuitous choice of coordinates helps solve

Form horizontal bonds into “sites” involving two types of atom
- “a” on the left end of a horizontal bond
- “b” on the right end
- all hoppings are between type $a$ and type $b$ atoms

Label sites by non-orthogonal coordinates $x_1$ and $x_2$
- axes at 30 degrees from horizontal
Hamiltonian

\[ H = K \sum_{x_1, x_2} a_{x_1, x_2}^\dagger b_{x_1, x_2} + b_{x_1, x_2}^\dagger a_{x_1, x_2} \]

\[ + a_{x_1+1, x_2}^\dagger b_{x_1, x_2} + b_{x_1-1, x_2}^\dagger a_{x_1, x_2} \]

\[ + a_{x_1, x_2-1}^\dagger b_{x_1, x_2} + b_{x_1, x_2+1}^\dagger a_{x_1, x_2} \]

- hops always between \( a \) and \( b \) sites

Go to momentum (reciprocal) space

- \( a_{x_1, x_2} = \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} e^{ip_1 x_1} e^{ip_2 x_2} \tilde{a}_{p_1, p_2} \)

- \(-\pi < p_{\mu} \leq \pi\)
Hamiltonian breaks into two by two blocks

\[ H = K \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \begin{pmatrix} \tilde{a}^\dagger_{p_1,p_2} & \tilde{b}^\dagger_{p_1,p_2} \end{pmatrix} \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}_{p_1,p_2} \\ \tilde{b}_{p_1,p_2} \end{pmatrix} \]

- where
  \[ z = 1 + e^{-ip_1} + e^{ip_2} \]

\[ \tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \]

Fermion energy levels at

\[ E(p_1, p_2) = \pm K|z| \]

- energy vanishes only when \(|z|\) does
- exactly two points \(p_1 = p_2 = \pm 2\pi/3\)
Topological stability

- contour of constant energy near a zero point
- phase of $z$ wraps around unit circle
- cannot collapse contour without going to $|z| = 0$

No band gap allowed

- Graphite is black and a conductor
No-go theorem Nielsen and Ninomiya

- periodicity of Brillouin zone
- wrapping around one zero must unwrap elsewhere
- two zeros is the minimum possible

Connection with chiral symmetry

- $b \rightarrow -b$ changes sign of $H$
- $\tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ anticommutes with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\sigma_3 \rightarrow \gamma_5$ in four dimensions
Four dimensions

Want Dirac operator $D$ to put into path integral action $\overline{\psi}D\psi$

- require “$\gamma_5$ Hermiticity”
  - $\gamma_5 D\gamma_5 = D^\dagger$
- work with Hermitean “Hamiltonian” $H = \gamma_5 D$
  - not the Hamiltonian of the 3D Minkowski theory

Require same form as the two dimensional case

$$\tilde{H}(p_\mu) = K \begin{pmatrix} 0 & \bar{z} \\ z^* & 0 \end{pmatrix}$$

- four component momentum, $(p_1, p_2, p_3, p_4)$
To keep topological argument

- extend \( z \) to quaternions
  \[ z = a_0 + i\vec{a} \cdot \vec{\sigma} \]
  - \(|z|^2 = \sum_\mu a_\mu^2\)

\(\tilde{H}(p_\mu)\) now a four by four matrix

- "energy" eigenvalues still \( E(p_\mu) = \pm K|z| \)
- constant energy surface topologically an \( S_3 \)
  - surrounding a zero should give non-trivial mapping
Implementation

- not unique
- here I follow Borici’s construction

Start with naive fermions

- forward hop between sites \( \gamma_\mu U \)
- backward hop between sites \(-\gamma_\mu U^\dagger\)
  - \( \mu \) is the direction of the hop
  - \( U \) is the usual gauge field matrix
- Dirac operator \( D \) anticommutes with \( \gamma_5 \)
  - an exact chiral symmetry
  - part of an exact \( SU(4) \times SU(4) \) chiral algebra

Karsten and Smit
In the free limit, solution in momentum space

\[ D(p) = 2i \sum_\mu \gamma_\mu \sin(p_\mu) \]

- for small momenta reduces to Dirac equation
- 15 extra Dirac equations for components of momenta near 0 or \( \pi \)

16 “Fermi points”
- “doublers”
Consider momenta maximally distant from the zeros: \( p_\mu = \pm \pi/2 \)

Select one of these points, i.e. \( p_\mu = +\pi/2 \) for every \( \mu \)

- \( D(p_\mu = \pi/2) = 2i \sum_\mu \gamma_\mu \equiv 4i \Gamma \)
- \( \Gamma \equiv \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \)
  - unitary, Hermitean, traceless 4 by 4 matrix
Now consider a unitary transformation

\[ \psi' (\mathbf{x}) = e^{-i\pi (x_1 + x_2 + x_3 + x_4)/2} \Gamma \psi (\mathbf{x}) \]

\[ \overline{\psi}' (\mathbf{x}) = e^{i\pi (x_1 + x_2 + x_3 + x_4)/2} \overline{\psi} (\mathbf{x}) \Gamma \]

- phases move Fermi points from \( p_\mu \in \{0, \pi\} \) to \( p_\mu \in \{\pm \pi/2\} \)
- \( \psi' \) uses new gamma matrices \( \gamma'_\mu = \Gamma \gamma_\mu \Gamma \)
- \( \Gamma = \frac{1}{2} (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) = \Gamma' \)
- new free action: \( \overline{D}(p) = 2i \sum_\mu \gamma'_\mu \sin(\pi/2 - p_\mu) \)

\( D \) and \( \overline{D} \) physically equivalent
Complimentarity: \[ D(p_\mu = \pi/2) = \overline{D}(p_\mu = 0) = 4i\Gamma \]

Combine the naive actions

\[ \mathcal{D} = D + \overline{D} - 4i\Gamma \]

Free theory

- \[ \mathcal{D}(p) = 2i \sum_\mu \left( \gamma_\mu \sin(p_\mu) + \gamma'_\mu \sin(\pi/2 - p_\mu) \right) - 4i\Gamma \]
- at \( p_\mu \sim 0 \) the \( 4i\Gamma \) term cancels \( \overline{D} \), leaving \( \mathcal{D}(p) \sim \gamma_\mu p_\mu \)
- at \( p_\mu \sim \pi/2 \) the \( 4i\Gamma \) term cancels \( D \), leaving \( \mathcal{D}(\pi/2 - p) \sim \gamma'_\mu p_\mu \)
  - Only these two zeros of \( \mathcal{D}(p) \) remain!
THEOREM: these are the only zeros of $D(p)$

- at other zeros of $D$, $\overline{D} - 4i\Gamma$ is large
- at other zeros of $\overline{D}$, $D - 4i\Gamma$ is large
Chiral symmetry remains exact

- \( \gamma_5 \mathcal{D} = -\mathcal{D} \gamma_5 \)
- \( e^{i\theta \gamma_5} \mathcal{D} e^{i\theta \gamma_5} = \mathcal{D} \)

But

- \( \gamma_5' = \Gamma \gamma_5 \Gamma = -\gamma_5 \)
- two species rotate oppositely
- symmetry is flavor non-singlet
Space time symmetries

- usual discrete translation symmetry
- \( \Gamma = \frac{1}{2} \sum_{\mu} \gamma_{\mu} \) treats primary hypercube diagonal specially
- action symmetric under subgroup of the hypercubic group
  - leaving this diagonal invariant
- includes \( Z_3 \) rotations amongst any three positive directions
  - \( V = \exp((i\pi/3)(\sigma_{12} + \sigma_{23} + \sigma_{31})/\sqrt{3}) \)
- cyclicly permutes \( x_1, x_2, x_3 \) axes
- physical rotation by \( 2\pi/3 \)

\[ \gamma_{\mu}, \gamma_{\nu} = 2i\sigma_{\mu\nu} \]
\[ [V, \Gamma] = 0 \]

\( V^3 = -1 \): we are dealing with fermions
Repeating with other axes generates the 12 element tetrahedral group

- subgroup of the full hypercubic group

Odd-parity transformations double the symmetry group to 24 elements

- \( V = \frac{1}{2\sqrt{2}}(1 + i\sigma_{15})(1 + i\sigma_{21})(1 + i\sigma_{52}) \)
  \([V, \Gamma] = 0\)
- permutes \(x_1, x_2\) axes
- \(\gamma_5 \rightarrow V^\dagger \gamma_5 V = -\gamma_5\)
Natural time axis along main diagonal $e_1 + e_2 + e_3 + e_4$

- $T$ exchanges the two Fermi points
- increases symmetry group to 48 elements

Karsten and Wilczek actions

- $e_4$ as the special direction

Charge conjugation: equivalent to particle hole symmetry

- $\mathcal{D}$ and $\mathcal{H} = \gamma_5 \mathcal{D}$ have eigenvalues in opposite sign pairs
Special treatment of main diagonal

- interactions can induce lattice distortions along this direction
  \[ \frac{1}{a}(\cos(ap) - 1)\bar{\psi}\Gamma\psi = O(a) \]
- symmetry restored in continuum limit
- at finite lattice spacing can tune
- coefficient of \( i\bar{\psi}\Gamma\psi \)
- 6 link plaquettes orthogonal to this diagonal
- zeros topologically robust under such distortions
- Nielsen Ninomiya, MC
Issues and questions

Requires a multiple of two flavors
  • can split degeneracies with Wilson terms

Only one exact chiral symmetry
  • not the full $SU(2) \otimes SU(2)$
    • enough to protect mass
    • $\pi^0$ a Goldstone boson
    • $\pi^\pm$ only approximate

Not unique
  • only need $z(p)$ with two zeros
  • above: Borici’s variation with orthogonal coordinates
    • alternatives: Karsten, Wilczek, MC
Comparison with staggered

- both have one exact chiral symmetry
- both have only approximate zero modes from topology
- four component versus one component fermion field
- two versus four flavors
- no uncontrolled extrapolation to two physical light flavors
Summary

- A strictly local lattice fermion action \( D(A) \)
- with one exact chiral symmetry \( \gamma_5 D = -D\gamma_5 \)
- describing two flavors; minimum required for chiral symmetry
- a linear combination of two “naive” fermion actions (Borici)

- Space-time symmetries
  - translations plus 48 element subgroup of hypercubic rotations
  - includes odd parity transformations
  - renormalization can induce anisotropy at finite \( a \)